Algebraic Differential Forms (Revised)

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1 Introduction

1.1 Motivation

Varieties play a major role in algebraic geometry. They are defined as the zero set of a collection of polynomials. Geometrically, this corresponds to the locus drawn out by the polynomials. Since varieties bear similarity to manifolds in the sense that there is interplay between algebra and geometry, it would be nice to collect some useful techniques from manifolds and apply it to varieties. One such notion is that of differential forms.

Formally, smooth differential 1-forms are smooth sections of the cotangent bundle. In other words, there is a smooth assignment of cotangent vectors for each point of the manifold. One can think of smooth differential 1-forms as a "differential operator" for functions on the manifolds. For the construction of 1-forms on varieties, we will mimic the more algebraic approach of thinking of differential forms as operators instead of the geometric picture of assigning cotangent vectors, even though there is indeed a notion of tangent space for varieties in textbooks such as [Sha12] and [SKKT00].

The resulting construct is a module, called the module of differential forms. In particular, it is universal in the sense that any other "differential operators" (called derivations in our case) factors through the module.

1.2 Preliminaries

The essay will make use of homological algebra / commutative algebra while developing the machinery. Since our motivation of the module comes from the cotangent bundle in manifolds, some basic knowledge on manifolds and varieties are needed, specifically that of tangent and cotangent spaces. Useful background knowledge on modules has been given a dedicated section in the appendix. While other references of commutative algebra and homological algebra can be found in [Eis07], [AM94] and [DF10]. As for the theory of manifolds and varieties, [Tu10] and [Sha12] respectively will suffice.

1.3 Objectives

The goal of this essay is to serve as an expository to basic results concerning the module of Kähler differentials. We will also see how good this cotangent bundle for varieties mimic that of manifolds. Some examples will also be illustrated showing that the module of Kähler differentials can be used to recover the cotangent space of the variety of a point, albeit somewhat convoluted.

Specifically, the second chapter delves into the heart of the essay: Derivations and the module of Kähler Differentials, as well as developing basic machinery to calculate the module such as the two exact sequences. The third chapter is a showcase / discussion of applications of the module. We will also see a construction of the module of Kähler differentials on coordinate rings. In the fourth and final chapter, we compare the module of Kähler Differentials with that of manifolds, and show that while it fails to become the "same" construct, we can still recover the cotangent space of varieties as classically defined in standard algebraic geometry textbooks.

2 Kähler Differentials

The goal of this section is to define the derivations and the module of Kähler differentials, as well as seeing some of its first consequences such as the two exact sequences. To show the existence of the module of Kähler differentials, we will see two different constructions of the module and then exhibit that they both satisfy the universal property.

2.1 Derivations

We begin with the definition of derivations. It will serve as the base of our discussions not only for the module of Kähler differentials, but also for manifolds.

By a ring, we mean that it is a commutative ring with identity $1 \neq 0$.

Definition 2.1.1 (Derivations)**.** Let A be a ring and B an A-algebra. Let M be a B-module. An A-derivation of B into M is an A-module homomorphism $d : B \to M$ such that the Leibniz rule holds:

$$d(b_1 b_2) = b_1 d(b_2) + d(b_1) b_2$$

for $b_1, b_2 \in B$. Denote the set of all A-derivations from B to M by

$$\mathrm{Der}_A(B, M) = \{d : B \to M \mid d \text{ is an } A \text{ derivation } \}$$

This is reminiscent of properties of a derivative. Indeed, from the above definition, take $A = \mathbb{R}$ and $B = M = \mathbb{R}[x_1, \ldots, x_n]$. Then the formal partial derivatives $\frac{\partial}{\partial x_i} : \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$ defined by

$$\left(f(x) = \sum_{k_1,\ldots,k_n} a_{k_1,\ldots,k_n} x_1^{k_1} \cdots x_i^{k_i} \cdots x_n^{k_n} \right) \mapsto \left(\frac{\partial f}{\partial x_i} = \sum_{k_1,\ldots,k_n} a_{k_1,\ldots,k_n} k_i x_1^{k_1} \cdots x_i^{k_i - 1} \cdots x_n^{k_n} \right)$$

(provided $k_i \geq 1$, otherwise the derivative is zero on that term) is $\mathbb{R}$-linear and satisfies the Leibniz rule. These are the two fundamental properties that a derivative should possess.

Recall that derivatives in calculus also satisfy the quotient rule and the fact that constant maps have zero derivatives. Instead of enforcing these requirements on the definition, we can show that the they can be derived from the consequences of d being linear and that it satisfies the Leibniz rule.

Lemma 2.1.2. *Let A be a ring and B an A-algebra Let M be a B-module. Let $d : B \to M$ be an A-derivation. Then $d(a) = 0$ for all $a \in A$.*

Proof. Since $d : B \to M$ is an A-module homomorphism, $d(a \cdot 1) = a \cdot d(1)$. We also have, by the Leibniz rule that $d(1) = 1 \cdot d(1) + d(1) \cdot 1 = 2d(1)$ which implies $d(1) = 0$. Thus $d(a \cdot 1) = a \cdot d(1) = 0$. □

The quotient rule is not so well defined in a general algebra. Indeed a ring does not necessarily have the notion of division and fractions. However recall that there is a systematic way of creating quotient elements in a ring. This is called localization.

Proposition 2.1.3. *Let B be an A-algebra. Let S be a multiplicative set of B. Let M be an $S^{-1}(B)$-module. Then for any A-derivation $d : B \to M$, there exists one unique way of extending the derivation to $d : S^{-1}B \to M$, defined by the formula:*

$$d\left(\frac{b}{s}\right) = \frac{sd(b) - bd(s)}{s^2}$$

Proof. Temporarily denote a derivation from $S^{-1}B$ to M by D. Suppose that $b \in B$ and $s \in S$. Notice that D has to satisfy the following:

$$d(b) = D(b) = D\left(s\frac{b}{s}\right) = \frac{b}{s}D(s) + sD\left(\frac{b}{s}\right)$$

Now multiply both sides by s^{-1} to obtain

$$D\left(\frac{b}{s}\right) = \frac{sD(b) - bD(s)}{s^2}$$

Thus any A-derivation $S^{-1}B$ to M must satisfy the above formula. This shows that there can only be one unique way of extending it.

For existence, we just have to show that it is a well defined map. Suppose that $\frac{a}{r} = \frac{b}{s}$. This means that there exists $q \in S$ such that $q(sa - rb) = 0$. The goal is to show that

$$\frac{rd(a) - ad(r)}{r^2} = \frac{sd(b) - bd(s)}{s^2}$$

or in other words, there exists $p \in S$ such that $p\left(s^2(rd(a) - ad(r)) - r^2 sd(b) - bd(s)\right) = 0$. I claim that $p = q^2$ does the job. Indeed we have that

$$\begin{aligned} q^2\left(s^2(rd(a) - ad(r)) - r^2 sd(b) - bd(s)\right) &= q^2(sad(rs) - rsd(as) - rbd(rs) + rsd(br)) \\ &= q^2((sa - rb)d(rs) + rs(d(br - as))) \\ &= rsq^2 d(br - as) \end{aligned}$$

Now in fact, $q^2 d(br - as) = 0$ because

$$\begin{aligned} q^2 d(br - as) &= q(qd(br - as)) \\ &= q(d(q(br - as)) - (br - as)d(q)) \\ &= 0 \end{aligned}$$

Thus we conclude. $\square$

We can see that $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}[x_1, \ldots, x_n], \mathbb{R}[x_1, \ldots, x_n])$ has more than just the standard partial derivatives from the module structure. For examples, the sum of partial derivatives

$$\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} : \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$$

defined by $f \mapsto \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_j}$. This is because of the extra structure of $\mathrm{Der}_A(B, M)$ as a B-module.

Lemma 2.1.4. *Let A be a ring and B an A-algebra. Let M be a B-module. Then $Der_A(B, M)$ is a B-module with the following operations:*

- *Addition is defined by sending $d_1, d_2 : B \to M$ to $(d_1 + d_2) : B \to M$ that maps b to $d_1(b) + d_2(b)$.*
- *The identity is given by the zero map $0 : B \to M$ given by $b \mapsto 0_M$.*
- *Left action is defined by $\cdot : B \times Der_A(B, M) \to Der_A(B, M)$ that sends $b \in B$ and $d : B \to M$ to $(bd) : B \to M$ defined by $u \mapsto b \cdot d(u)$.*

Proof. Firstly, $\mathrm{Der}_A(B, M)$ is an abelian group. We check the group axioms.

- Closure: Let $a \in A$ and $b_1, b_2 \in B$. $d_1 + d_2 : B \to M$ is an A-module homomorphism because

  $$\begin{aligned} (d_1 + d_2)(ab_1 + b_2) &= d_1(ab_1 + b_2) + d_2(ab_1 + b_2) \\ &= ad_1(b_1) + d_1(b_2) + ad_2(b_1) + d_2(b_2) \\ &= a(d_1 + d_2)(b_1) + (d_1 + d_2)(b_2) \end{aligned}$$

  Finally, the Leibniz rule is satisfied because

  $$\begin{aligned} (d_1 + d_2)(b_1 b_2) &= d_1(b_1 b_2) + d_2(b_1 b_2) \\ &= b_1 d_1(b_2) + d_1(b_1)b_2 + b_1 d_2(b_2) + d_2(b_1)b_2 \\ &= b_1(d_1 + d_2)(b_2) + (d_1 + d_2)(b_1)b_2 \end{aligned}$$

- Associativity: Follows from the fact that M is a group
- Identity: The zero map is the identity since for any $d : B \to M$, $d + 0 : B \to M$ sends b to $d(b)$ and thus $d + 0 = d$.
- Inverse: For each $d : B \to M$ the maps sending b to $-d(b)$ is an inverse

- Abelian: Follows from the fact that M is abelian.

Finally, left action is defined by $\cdot : B \times \mathrm{Der}_A(B, M) \to \mathrm{Der}_A(B, M)$ that sends $b \in B$ and $d : B \to M$ to $(bd) : B \to M$ defined by $u \mapsto b \cdot d(u)$. Associativity and identity is clear. $\square$

However, second order derivatives (which are compositions of the first order partial derivatives) are not derivations! Indeed they satisfy not the Leibniz property but instead, we have that

$$\frac{\partial(fg)}{\partial x_i x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j} g + f \frac{\partial g}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i x_j} + \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} + \frac{\partial^2 g}{\partial x_i x_j}$$

which is in general a more complicated identity than the Leibniz rule.

2.2 Kähler Differentials

We now define the module of Kähler Differentials which is the main object of study. For each A-derivation d from an A-algebra B to a B-module M, d factors through a universal object no matter what d we choose. This is the content of the following definition.

Definition 2.2.1. Kähler Differentials Let A be a ring and let B be an A-algebra. A B-module $\Omega^1_{B/A}$ together with an A-derivation $d : B \to \Omega^1_{B/A}$ is said to be a module Kähler Differentials of B over A if it satisfies the following universal property:

For any B-module M, and for any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega^1_{B/A} \to M$ such that $d' = f \circ d$. In other words, the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega^1_{B/A} \arrow[d, dashed, "\exists! f"] \\ & M \end{tikzcd}$$

The following lemma restates the universal property in a more categorical way.

Lemma 2.2.2. *Let A be a ring and B an A-algebra. Let M be a B-module. Then there is a canonical B-module isomorphism*

$$Hom_B(\Omega^1_{B/A}, M) \cong Der_A(B, M)$$

defined via the universal property of the module of Kähler Differentials.

Proof. Fix M a B-module. Let $d' \in \mathrm{Der}_A(B, M)$. By the universal property of $\Omega^1_{B/A}(M)$, there exists a unique B-module homomorphism $f : \Omega^1_{B/A} \to M$ such that $d' = f \circ d$. This gives a map $\phi : \mathrm{Der}_A(B, M) \to \mathrm{Hom}_B(\Omega^1_{B/A}, M)$ defined by $\phi(d') = f$.
Conversely, given a map $g \in \mathrm{Hom}_B(\Omega^1_{B/A}, M)$, pre-composition with d gives a pull back map $d^* : \mathrm{Hom}_B(\Omega^1_{B/A}, M) \to \mathrm{Der}_A(B, M)$ defined by $d^*(g) = g \circ d$. These map are inverses of each other:

$$\begin{aligned} (d^* \circ \phi)(d') &= d^*(f) \\ &= f \circ d \\ &= d' && \text{(By universal property)} \end{aligned}$$

and $(\phi \circ d^*)(g) = \phi(g \circ d) = g$. Thus these map is a bijective map of sets.

It remains to show that d^* is a B-module homomorphism. Let $f, g \in \mathrm{Hom}_B(\Omega^1_{B/A}, M)$.

- $d^*(f + g) = (f + g) \circ d$ is a map

  $$b \overset{d}{\mapsto} d(b) \overset{f+g}{\mapsto} f(d(b)) + g(d(b))$$

  for $b \in B$. $d^*(f) + d^*(g) = f \circ d + g \circ d$ is a map

  $$b \mapsto f(d(b)) + g(d(b))$$

  thus addition is preserved by d^*.

- Let $u \in B$. We want to show that $d^*(u \cdot f) = u \cdot d^*(f)$. The left hand side sends an element $b \in B$ by
$$b \overset{d}{\mapsto} d(b) \overset{u \cdot f}{\mapsto} u \cdot f(d(b))$$
The right hand side sends $b \mapsto u \cdot f(d(b))$. Thus proving they are the same.

And so we have reached the conclusion. $\square$

As any category theorist will realize, this is almost the same saying the functor $\mathrm{Der}_A(B, -) : {}_B\mathbf{Mod} \to {}_B\mathbf{Mod}$ is representable via the module of Kähler Differentials. But of course we neither demonstrated that $\mathrm{Der}_A(B, -)$ is a functor, nor the fact that the above isomorphism is natural in M. For our purposes, the above lemma and its content will suffice.

As always, such a definition via the universal property does not show the existence of $\Omega^1_{B/A}$ for any appropriate choice of A, B. In the following, we shall demonstrate two different constructions of the module with two different purposes.

Proposition 2.2.3. *Let A be a ring and B be an A-algebra. Let F be the free B-module generated by the symbols $\{d(b) \mid b \in B\}$. Let R be the submodule of F generated by the following relations:*

- $d(a_1b_1 + a_2b_2) - a_1d(b_1) - a_2d(b_2)$ *for all $b_1, b_2 \in B$ and $a_1, a_2 \in A$*
- $d(b_1b_2) - b_1d(b_2) - b_2d(b_1)$ *for all $b_1, b_2 \in B$*

Then F/R is a module of Kähler Differentials for B over A.

Proof. Clearly F/R is a B-module. Moreover, define $d : B \to F/R$ by $b \mapsto d(b) + R$. This map is an A-derivation since the following are satisfied:

- d is an A-module homomorphism: Let $b_1, b_2 \in B$ and $a_1, a_2 \in A$. Then $a_1b_1 + a_2b_2$ is mapped to $d(a_1b_1 + a_2b_2) + R$. We know from the relations that $d(a_1b_1 + a_2b_2) + R = a_1d(b_1) + a_2d(b_2) + R$. Thus d is A-linear.
- d satisfies the Leibniz rule: Let $b_1, b_2 \in B$. Then b_1b_2 is mapped to $d(b_1b_2) + R$. Since $d(b_1b_2) + R = b_1d(b_2) + d(b_1)b_2$, we have that b_1b_2 is mapped to $b_1d(b_2) + d(b_1)b_2 + R$.

This shows that $d : B \to F/R$ is an A derivation.

It remains to show that $(F/R, d)$ has the universal property. Let M be a B-module and $d' : B \to M$ an A-derivation. Define a map $f : F \to M$ on generators by $d(b) \mapsto d'(b)$ and extending from generators to the entire module. This is a B-module homomorphism by definition. Clearly $f \circ d = d'$. It also unique since f is defined on the generators of F.

Finally we want to show that f projects to a map $\bar{f} : F/R \to M$. This requires us to check that $f(d(a_1b_1 + a_2b_2)) = f(a_1d(b_1) + a_2d(b_2))$ and $f(d(b_1b_2)) = f(b_1d(b_2) + d(b_1)b_2)$. But this is clear. Since $f : F \to R$ is a B-module homomorphism, we have
$$f(d(a_1b_1 + a_2b_2)) - f(a_1d(b_1) + a_2d(b_2)) = 0$$
and
$$f(d(b_1b_2)) - f(b_1d(b_2) + d(b_1)b_2) = 0$$
implying f sends $d(a_1b_1 + a_2b_2) - a_1d(b_1) - a_2d(b_2)$ and $d(b_1b_2) - b_1d(b_2) - d(b_1)b_2$ to 0. Since we checked them on generators of R this result extends to all of R. Thus we are done. $\square$

Aside from the construction through quotients, we can also express the module explicitly via the kernel of a diagonal morphism. Using the universal property, we see that all these constructions are the same.

Proposition 2.2.4. *Let A be a ring and B be an A-algebra. Let $f : B \otimes_A B \to B$ be a function defined to be $f(b_1 \otimes_A b_2) = b_1b_2$. Let I be the kernel of f. Then $(I/I^2, d)$ is a module of Kähler Differentials of B over A, where the derivation is the homomorphism $d : B \to I/I^2$ defined by $db = 1 \otimes b - b \otimes 1 \pmod{I^2}$.*

Proof. We break down the proof in 3 main steps.
Step 1: Show that $\ker(f) = \langle 1 \otimes b - b \otimes 1 \mid b \in B \rangle$.
Write $I = \langle 1 \otimes b - b \otimes 1 \mid b \in B \rangle$. For any generator $1 \otimes b - b \otimes 1$ of I, we see that

$$f(1 \otimes b - b \otimes 1) = 0$$

Thus $I \subseteq \ker(f)$. Now suppose that $\sum_{i,j} b_i \otimes b_j \in \ker(f)$. Then using the identity

$$b_i \otimes b_j = b_i b_j \otimes 1 + (b_i \otimes 1)(1 \otimes b_j - b_j \otimes 1)$$

and the fact that $b_i b_j = 0$ (because $0 = f(b_i \otimes b_j) = b_i b_j$) we see that

$$\sum_{i,j} b_i \otimes b_j = \sum_{i,j} (b_i \otimes 1)(1 \otimes b_j - b_j \otimes 1)$$

Since each $1 \otimes b_j - b_j \otimes 1$ lies in $\ker(f)$, we conclude that $\sum_{i,j} b_i \otimes b_j$ so that $I = \ker(f)$.
Step 2: Check that $d : B \to I/I^2$ is an A-derivation.

- $d : B \to I/I^2$ is an A-module homomorphism: Let $a_1 a_2 \in A$ and $b_1, b_2 \in B$. Then we have

$$\begin{aligned} d(a_1 b_1 + a_2 b_2) &= 1 \otimes (a_1 b_2 + a_2 b_2) - (a_1 b_2 + a_2 b_2) \otimes 1 + I^2 \\ &= a_1(1 \otimes b_1) + a_2(1 \otimes b_2) - a_1(b_1 \otimes 1) - a_2(b_2 \otimes 1) + I^2 \\ &= a_1 d(b_1 b_2) + a_2 d(b_1 b_2) + I^2 \end{aligned}$$

  Thus we are done. (Notice that we did not use the fact that all the expressions are taken modulo I^2)

- d satisfies the Leibniz rule: Let $b_1, b_2 \in B$. Then we have $d(b_1 b_2) = 1 \otimes b_1 b_2 - b_1 b_2 \otimes 1 + I^2$ on one hand. On the other hand we have

$$b_1 d(b_2) + b_2 d(b_1) = b_1(1 \otimes b_2 - b_2 \otimes 1) + b_2(1 \otimes b_1 - b_1 \otimes 1) + I^2$$

  Subtracting them gives

$$\begin{aligned} d(b_1 b_2) - b_1 d(b_2) - b_2 d(b_1) &= 1 \otimes b_1 b_2 - b_1 \otimes b_2 - b_2 \otimes b_1 + b_2 b_1 \otimes 1 \\ &= (1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1) + I^2 \end{aligned}$$

  But $(1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1)$ lies in I^2 thus subtraction gives 0.

Thus d is an A-derivation.

Step 3: Show that the universal property is satisfied.
Let M be a B-module and $d' : B \to M$ an A-derivation. We want to find a unique $\tilde{\phi} : B \to M$ such that $d' = \tilde{\phi} \circ d$.

Step 3.1: Construct a homomorphism of A-algebra from $B \otimes B$ to $B \ltimes M$
Define $\phi : B \otimes B \to B \ltimes M$ (Refer to 7.1.7 for definition of $B \ltimes M$) by

$$\phi(b_1 \otimes b_2) = (b_1 b_2, b_1 d'(b_2))$$

and extend it linearly so that $\phi(b_1 \otimes b_2 + b_3 \otimes b_4) = \phi(b_1 \otimes b_2) + \phi(b_3 \otimes b_4)$. This is a homomorphism of A-algebra since

- Addition is preserved: This is by definition.
- $\phi(ab_1 \otimes b_2) = \phi(b_1 \otimes ab_2) = a\phi(b_1 \otimes b_2)$: Let $a \in A$ and $b_1 \otimes b_2 \in B \otimes_A B$. Then

$$\begin{aligned} \phi(ab_1 \otimes b_2) &= (ab_1 b_2, ab_1 d'(b_2)) \\ &= a \cdot \phi(b_1 \otimes b_2) \\ \phi(b_1 \otimes ab_2) &= (ab_1 b_2, b_1 d'(ab_2)) \\ &= (ab_1 b_2, ab_1 d'(b_2)) \end{aligned}$$

  Thus we are done.

- Product is preserved: For $u_1, u_2, v_1, v_2 \in B$, we have

$$\begin{aligned}\phi((u_1 \otimes u_2) \cdot \phi(v_1 \otimes v_2)) &= (u_1u_2, u_1d'(u_2)) \cdot (v_1v_2, v_1d'(v_2)) \\ &= (u_1u_2v_1v_2, u_1u_2v_1d'(v_2) + v_1v_2u_1d'(u_2)) \\ &= (u_1v_1u_2v_2, u_1v_1d'(u_2v_2)) \\ &= \phi(u_1v_1 \otimes u_2v_2)\end{aligned}$$

Thus ϕ is a homomorphism of A-algebra.

Step 3.2: Construct $\tilde{\phi}$ from ϕ.
Since ϕ is a map $B \otimes B$ to $B \ltimes M$, we can restrict this map to I a result in a new map $\bar{\phi} : I \to B \ltimes M$. Notice that for $1 \otimes b - b \otimes 1$ a generator of I, we have

$$\begin{aligned}\bar{\phi}(1 \otimes b - b \otimes 1) &= \bar{\phi}(1 \otimes b) - \bar{\phi}(b \otimes 1) \\ &= (b, d'(b)) - (b, d'(1)) \\ &= (b, d'(b)) - (b, 0) \\ &= (0, d'(b))\end{aligned}$$

Thus we actually have a map $\bar{\phi} : I \to M$. Finally, notice that for $(1 \otimes u - u \otimes 1)(1 \otimes v - v \otimes 1)$ a generator of I^2, we have

$$\begin{aligned}\bar{\phi}(x) &= \phi(1 \otimes u - u \otimes 1)\phi(1 \otimes v - v \otimes 1) \\ &= \sum (0, d'(u))(0, d'(v)) \\ &= \sum (0, 0) && \text{(Mult. in Trivial Extension)} \\ &= (0, 0)\end{aligned}$$

which shows $\bar{\phi}$ kills of I^2 and thus $\bar{\phi}$ factors through I/I^2 so that we get a map $\tilde{\phi} : I/I^2 \to M$.

Step 3.3: Show that $\tilde{\phi}$ satisfies all the required properties.
For $b \in B$, we have that

$$\tilde{\phi}(d(b)) = \tilde{\phi}(1 \otimes b - b \otimes 1 + I^2) = d'(b)$$

and thus $d' = \tilde{\phi} \circ d$. Moreover, this map is unique since it is defined on the generators of I, namely the $d(b)$ for $b \in B$.

This concludes the proof.
Materials referenced: [Vak22], [Kun86], [Mat80] □

Despite being a more convoluted way to construct the module of Kähler Differentials, it turns out that the advantage of this construction is that it generalizes well to the theory of schemes. Interested readers are referred to [Har77].

Our first step towards computing the module of Kähler Differentials for coordinate rings comes from a computation of the polynomial ring.

Lemma 2.2.5. *Let A be a ring and $B = A[x_1, \ldots, x_n]$ so that B is an A-algebra. Then*

$$\Omega^1_{B/A} \cong \bigoplus_{i=1}^{n} Bd(x_i)$$

In particular, the module $\Omega^1_{B/A}$ is a finitely generated B-module.

Proof. The fact that $\Omega^1_{B/A}$ is finitely generated directly follows from the claimed isomorphism. So let us prove the isomorphism. Consider the map $p : B \to \bigoplus_{k=1}^{n} Bd(x_k)$ given by

$$f \mapsto \left(\frac{\partial f}{\partial x_1} d(x_1), \ldots, \frac{\partial f}{\partial x_n} d(x_n)\right)$$

where each partial derivative here is the formal derivative of f with respect to the variable. Now that by choosing the function $x_k \in B$, we have $p(x_k) = (0, \ldots, 0, d(x_k), 0, \ldots, 0)$. Since each partial derivative is B-linear and satisfies the Leibniz rule, the universal property of $\Omega^1_{B/A}$ implies that there exists a unique B-linear map $\phi : \Omega^1_{B/A} \to \bigoplus_{k=1}^n Bd(x_k)$ that satisfies $p = \phi \circ d$.

I claim that the map $\psi : \bigoplus_{k=1}^n Bd(x_k) \to \Omega^1_{B/A}$ defined by

$$(g_1, \ldots, g_n) \mapsto \sum_{k=1}^n g_k d(x_k)$$

is the inverse of ϕ. Notice that this makes sense because here we think of $\Omega^1_{B/A}$ as being generated by $d(b)$ for all $b \in B$. It is evident that this map is B-linear. Now we have that

$$\begin{aligned}
\phi(\psi(g_1 d(x_1), \ldots, g_n d(x_n))) &= \phi\left(\sum_{k=1}^n g_k d(x_k)\right) \\
&= \sum_{k=1}^n g_k \phi(d(x_k)) && (\phi \text{ is } B\text{-linear}) \\
&= \sum_{k=1}^n g_k p(x_k) \\
&= \sum_{k=1}^n g_k (0, \ldots, 0, d(x_k), \ldots, 0, \ldots, 0) \\
&= (g_1 d(x_1), \ldots, g_n d(x_n))
\end{aligned}$$

Now since $\Omega^1_{B/A}$ is generated by the symbols $d(f)$, an arbitrary element of $\Omega^1_{B/A}$ is given by $\sum_{k=1}^n u_k d(f_k)$ for some $u_k, f_k \in B$. Since ϕ and ψ are B-linear, it suffices to prove that $\psi(\phi(d(f))) = d(f)$. We have that

$$\begin{aligned}
\psi(\phi(d(f))) &= \psi(p(f)) \\
&= \psi\left(\frac{\partial f}{\partial x_1} d(x_1), \ldots, \frac{\partial f}{\partial x_n} d(x_n)\right) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x_k} d(x_k)
\end{aligned}$$

The problem remains to show that $d(f) = \sum_{k=1}^n \frac{\partial f}{\partial x_k} d(x_k)$. But this formula is true because d is B-linear and satisfies the Leibniz rule. Thus ϕ and ψ are mutual inverses so that we obtain the desired isomorphism. Materials referenced: [Eis07], [Pro11], [GW23] □

Notice that throughout all of our definitions, there is not a single place where we have to define genuine limits similar to that in analysis or calculus. Instead, we start with some algebraic objects such as rings, algebras and module, bestowed maps between them with $\mathbb{R}$-linearity and Leibniz rule, and we ended up in a situation in analysis / calculus. it shows that we have captured the algebraic properties of derivatives in the sense of calculus and are able to reproduce it here.

2.3 Transfering the System of Differentials

This section aims to develop the necessary machinery in order to compute the module of Kähler Differentials for coordinate rings. We will see explicit calculation of the cuspidal cubic, an ellipse and the double cone to demonstrate how the two exact sequences can be used along with the Jacobian of the defining equations of the variety to compute the module of Kähler Differentials.

Theorem 2.3.1. *First Exact Sequence Let B, C be A-algebras and let $\phi : B \to C$ be an A-algebra homomorphism. Then the following sequence is an exact sequence of C-modules:*

$$\Omega^1_{B/A} \otimes_B C \xrightarrow{f} \Omega^1_{C/A} \xrightarrow{g} \Omega^1_{C/B} \longrightarrow 0$$

where f and g is defined respectively as

$$f(d_{B/A}(b) \otimes c) = c \cdot d_{C/A}(\phi(b))$$

and

$$g(d_{C/A}(c)) = d_{C/B}(c)$$

and extended linearly.

Proof. Denote $d_{B/A}, d_{C/A}, d_{C/B}$ the derivations for $\Omega^1_{B/A}, \Omega^1_{C/A}, \Omega^1_{C/B}$ respectively. Clearly g is surjective since for any $c_1 d_{C/B}(c_2) \in \Omega^1_{C/B}$, just choose $c_1 d_{C/A}(c_2) \in \Omega^1_{C/A}$. We just have to show that $\ker(g) = \operatorname{im}(f)$. It is enough to show that

$$0 \longrightarrow \operatorname{Hom}_C(\Omega^1_{C/B}, N) \xrightarrow{g^*} \operatorname{Hom}_C(\Omega^1_{C/A}, N) \xrightarrow{f^*} \operatorname{Hom}_C(\Omega^1_{B/A} \otimes_B C, N)$$

is exact by 7.1.2. Using the fact that $\operatorname{Hom}_C(\Omega^1_{B/A} \otimes_B C, N) \cong \operatorname{Hom}_B(\Omega^1_{B/A}, N)$ (7.1.3) and the fact that $\operatorname{Hom}(\Omega^1_{B/A}, N) \cong \operatorname{Der}_A(B, N)$, we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_C(\Omega^1_{C/B}, N) \arrow[r, "g^*"] & \operatorname{Hom}_C(\Omega^1_{C/A}, N) \arrow[r, "f^*"] \arrow[dd, shift right, "\cong"'] & \operatorname{Hom}_C(\Omega^1_{B/A} \otimes_B C, N) \arrow[d, shift right, "\cong"'] \\
 & & & \operatorname{Hom}_B(\Omega^1_{B/A}, N) \arrow[u, shift right] \arrow[d, dashed, "\cong"] \\
0 \arrow[r] & \operatorname{Der}_B(C, N) \arrow[r, "u"] \arrow[uu, "\cong"] & \operatorname{Der}_A(C, N) \arrow[r, "v"] \arrow[uu, shift right] & \operatorname{Der}_A(B, N)
\end{tikzcd}$$

where u and v are obtained by going along the appropriate commutative squares. Now explicitly, we have the following:

- The map u is actually an inclusion. To see this, let us trace an element $h \in \operatorname{Der}_A(C, N)$. Under the far left vertical isomorphism, h is send to the unique map $\eta : \Omega^1_{C/B} to N$ such that $h = \eta \circ d_{C/B}$. Precomposing with g^* gives the map $\eta \circ g : \Omega^1_{C/A} \to N$. Then under the middle vertical isomorphism, we obtain a unique map $k : C \to N$ such that $k = (\eta \circ g) \circ d_{C/A}$. But notice that

$$k = \eta \circ (g \circ d_{C/A}) = \eta \circ d_{C/B} = h$$

  so $k = h$ and u is indeed an inclusion.

- The map v is actually a restriction of scalars. To see this, let us trace an element $y \in \operatorname{Der}_A(C, N)$. Under the middle vertical isomorphism, y is sent to the unique map $\psi : \Omega^1_{C/A} \to N$ such that $y = \psi \circ d_{C/A}$. Precomposing with f^* gives the map $\psi \circ f : \Omega^1_{B/A} \otimes_B C \to N$. Under the top right isomorphism, we notice that we can think of it as there is a B-linear inclusion $\iota : \Omega^1_{B/A} \hookrightarrow \Omega^1_{B/A} \otimes C$ defined by $d(b) \mapsto d(b) \otimes 1_C$. Let us the latter approach. Because ι is B-linear, the map $\psi \circ f \circ \iota$ is also B-linear. By the universal property of $\Omega^1_{B/A}$, we obtain a unique map $z : B \to N$ such that $z = \psi \circ f \circ \iota \circ d_{B/A}$. I claim that the following diagram (consider all module homomorphisms as B-linear) commutes:

$$\begin{tikzcd}
B \arrow[r, "d_{B/A}"] \arrow[dr, "\phi"'] & \Omega^1_{B/A} \arrow[r, hook, "\iota"] \arrow[d, dashed, "\exists ! \epsilon"'] & \Omega^1_{B/A} \otimes C \arrow[d, "f"] \\
 & C \arrow[r, "d_{C/A}"] \arrow[dr, "y"] & \Omega^1_{C/A} \arrow[d, dashed, "\exists ! \psi"] \\
 & & N
\end{tikzcd}$$

  Now the top left and bottom right triangles commute by the universal property of the module of Kähler Differentials. Let $d_{B/A}(b) \in \Omega^1_{B/A}$. On one hand we have that $(d_{C/A} \circ \epsilon)(d_{B/A}(b)) = d_{C/A}(\phi(b)) = f(d_{B/A}(b) \otimes 1_C)$. On the other hand we have that $f(\iota(d_{B/A}(b))) = f(d_{B/A}(b) \otimes 1_C)$ and so the top right square commutes. Thus the diagram commutes. Now I want to show that $z = y \circ \phi$. Now this is why maths is beautiful:

$$z = \psi \circ f \circ \iota \circ d_{B/A} = \psi \circ d_{C/A} \circ \epsilon \circ d_{B/A} = \psi \circ d_{C/A} \circ \phi = y \circ \phi$$

  This shows that the map v sending y to $z = y \circ \phi$ is indeed a restriction of scalars!

Now let $h \in \mathrm{im}(u)$. Then h is B-linear (u is just an inclusion). Thus $\mathrm{im}(u) \subseteq \ker(v)$. Now let $y \in \ker(v)$. Then $y \circ \phi = 0$ (v is just a restriction). I claim that y is B-linear. This is done by considering the structure of C as a B-module. The module structure $\cdot : B \times C \to C$ is given by $b \cdot c = \phi(b)c$ and this is similar for N. So we have

$$y(b \cdot c) = y(\phi(b)c) = y(\phi(b))c + \phi(b)y(c) = \phi(b)y(c) = b \cdot y(c)$$

Thus y is B-linear. We conclude that $\mathrm{im}(u) = \ker(v)$ so that the first exact sequence is indeed exact. $\square$

Theorem 2.3.2. *Second Exact Sequence Let A be a ring and B an A-algebra. Let I be an ideal of B and $C = B/I$. Then the following sequence is an exact sequence of C-modules:*

$$I/I^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B C \xrightarrow{f} \Omega^1_{C/A} \longrightarrow 0$$

where δ and f is defined respectively as

$$\delta(i + I^2) = d(i) \otimes 1$$

and

$$f(d(b) \otimes c) = c \cdot d(\phi(b))$$

and then extended linearly.

Proof. Notice that δ is well defined. Indeed, if $i + I^2 = j + I^2$, then there exists $h_1, h_2 \in I$ such that $i - j = h_1 h_2$. Now we have that

$$\begin{aligned}
\delta(i - j) &= d(h_1 h_2) \otimes 1 \\
&= h_1 d(h_2) \otimes 1 + h_2 d(h_1) \otimes 1 \\
&= d(h_2) \otimes h_1 + I + d(h_1) \otimes h_2 + I \\
&= d(h_2) \otimes 0 + d(h_1) \otimes 0 \\
&= 0
\end{aligned}$$

We can see that f is surjective. Indeed for any $d(b + I) \in \Omega^1_{C/A}$, just choose $d(b) \otimes 1 \in \Omega^1_{B/A} \otimes_B C$. Then $f(d(b) \otimes 1) = d(b + I)$.

It remains to show that $\mathrm{im}(\delta) = \ker(f)$. Notice that to prove the exactness of the sequence in question, we just have to show the exactness of the following sequence (by 7.1.2):

$$0 \longrightarrow \mathrm{Hom}_C(\Omega^1_{C/A}, N) \xrightarrow{f^*} \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B \tfrac{B}{I}) \xrightarrow{\delta^*} \mathrm{Hom}_C(I/I^2, N)$$

Using the fact that $I/I^2 \cong I \otimes_B \frac{B}{I}$ (by 7.1.4) and $\mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B B/I, N) = \mathrm{Hom}_B(\Omega^1_{B/A}, N)$ (by 7.1.3) we can transform this sequence into

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_C(\Omega^1_{C/A}, N) \arrow[r, "f^*"] & \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B \frac{B}{I}, N) \arrow[r, "\delta^*"] \arrow[d, shift right, "\cong"'] & \mathrm{Hom}_C(\frac{I}{I^2}, N) \arrow[d, "\cong"'] \\
 & & \mathrm{Hom}_B(\Omega^1_{B/A}, N) \arrow[u, shift right] \arrow[dd, shift right, "\cong"'] & \mathrm{Hom}_C(I \otimes_B \frac{B}{I}, N) \arrow[d, "\cong"'] \\
 & & & \mathrm{Hom}_B(I, N) \arrow[d, "\cong"'] \\
0 \arrow[r] & \mathrm{Der}_A(C, N) \arrow[uuu, "\cong"] \arrow[r, "v"] & \mathrm{Der}_A(B, N) \arrow[uu, shift right] \arrow[r, "w"] & \mathrm{Der}_A(I, N)
\end{tikzcd}$$

We need to show exactness between on $\mathrm{Hom}_B(\Omega^1_{B/A}, N)$. u and v here are the given by their corresponding commuting squares. Explicitly, we have that

- Since this sequence is derived directly from the first exact sequence, we note that the map v is exactly the map v in the proof of the first exact sequence. Hence v is just a restriction of scalars.
- By employing the same strategy, one can show that the map w is also a restriction of scalars from B to I.

Now we want to show that $\mathrm{im}(v) = \ker(w)$. Let $h \in \mathrm{im}(v)$. Then there exists $\bar{h} \in \mathrm{Der}_A(B/I, N)$ such that $h = \bar{h} \circ p$. Now $w(h) = h \circ \iota = \bar{h} \circ p \circ \iota$ where $\iota : I \to B$ is the inclusion. We have that $p \circ \iota$ is the 0 map hence $w(h) = 0$ and so $\mathrm{im}(v) \subseteq \ker(w)$. Now let $y \in \ker(w)$. Then $y(i) = 0$ for all $i \in I$. By the universal property of quotients, there exists a unique map $\bar{y} : B/I \to N$ such that $y = \bar{y} \circ p$. Thus $y \in \operatorname{im}(v)$. We conclude that $\operatorname{im}(v) = \ker(w)$. □

A very nice application towards computing the module of differential forms is given by the second exact sequence. For $B = A[x_1, \dots, x_n]$ and $C = \frac{B}{I=(f_1,\dots,f_r)}$, we can use 7.1.5 and 2.2.5 to see that $\Omega^1_{B/A} \otimes C \cong \bigoplus_{i=1}^n Cdx_i$. By the second exact sequence 2.3.2, we see that

$$\Omega^1_{C/A} \cong \operatorname{coker}\left(\frac{I}{I^2} \to \bigoplus_{i=1}^n Cdx_i\right)$$

Since I/I^2 is a C-module, by 7.1.6 there exists a surjective map $\bigoplus_{i=1}^m Cde_i \twoheadrightarrow I/I^2$. In fact $m = r$ since I is finitely generated by $f_1, \dots, f_r$ and thus the map sends e_i to f_i for $1 \le i \le r$.

Now consider the map

$$J : \bigoplus_{i=1}^r Cde_i \twoheadrightarrow \frac{I}{I^2} \to \bigoplus_{i=1}^n Cdx_i$$

This is a map from a free module of rank r to a free module of rank n. So we can write this in an $n \times r$ matrix. Since the map $I/I^2 \to \bigoplus_{i=1}^n Cdx_i$ sends f_i to $d(f_i) = \sum_{k=1}^n \frac{\partial f_i}{\partial x_k} dx_k$ (by second exact sequence 2.3.2) and e_i is sent f_i, we have that J is the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}$$

Finally, since $\operatorname{im}(A \twoheadrightarrow B \to C) = \operatorname{im}(B \to C)$, we thus have

$$\operatorname{coker}(J) \cong \Omega^1_{C/A}$$

which means that $\Omega^1_{C/A}$ is just the cokernel of the matrix. This exposition can be found in [Eis07].

This leads to our first calculations of the module of Kähler Differentials.

Example 2.3.3. Cuspidal Cubic: Part 1 Write $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$ the vanishing locus of the cuspidal cubic. Then the module of Kähler differentials $\Omega^1_{\mathbb{C}[V]/\mathbb{C}}$ can be calculated using the above method of the cokernel. An easy calculation shows that J is the matrix $\begin{pmatrix} -3x^2 \\ 2y \end{pmatrix}$. So the image of J is $(-3x^2)dx \oplus (2y)dy$ and thus

$$\Omega^1_{\mathbb{C}[V]/\mathbb{C}} \cong \frac{\mathbb{C}[V]dx \oplus \mathbb{C}[V]dy}{((-3x^2)dx \oplus (2y)dy)}$$

Example 2.3.4. Ellipse: Part 1 Write $W = \mathbb{V}(4x^2 + 9y^2 - 36) \subseteq \mathbb{A}^2_{\mathbb{C}}$ the vanishing locus of the ellipse. Similar to the previous example, it is easy to see that

$$\Omega^1_{(\mathbb{C}[W])/\mathbb{C}} \cong \frac{\mathbb{C}[W]dx \oplus \mathbb{C}[W]dy}{((8x)dx \oplus (18y)dy)}$$

Example 2.3.5. The Double Cone: Part 1 Write $U = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{A}^3_{\mathbb{C}}$ for the vanishing locus of the double cone. Again we can show that

$$\Omega^1_{\mathbb{C}[U]/\mathbb{C}} \cong \frac{\mathbb{C}[U]dx \oplus \mathbb{C}[U]dy \oplus \mathbb{C}[U]dz}{(2xdx \oplus 2ydy \oplus -2zdz)}$$

using the fact that the Jacobian matrix of the equation of the double cone is given by

$$J = \begin{pmatrix} 2x & 2y & (-2z) \end{pmatrix}^T$$

3 Applications of the Module of Kähler Differentials

3.1 Characterization for Separability

Recall that we say that a field F/K is separable if the minimal polynomial of all $\alpha \in K$ has no repeated roots in any splitting fields. There is also a close connection between separability and formal derivatives because by taking derivatives we can detect whether a root has multiplicity ≥ 2.

The module of Kähler differentials give a necessary and sufficient condition for a finite extension to be separable. But before the main proposition, we will need a lemma.

Lemma 3.1.1. *Let L/K be a finite field extension and $\Omega^1_{L/K}$ the module of Kähler Differentials. Let $f(b) = c_0 + c_1 b + \cdots + c_n b^n \in L$ for $c_0, \ldots, c_n \in K$ and $b \in L$. Then $d(f(b)) = f'(b)d(b)$ where $f'(b)$ is the derivative of $f(b)$ with respect to b in the sense of calculus.*

Proof. Since $f(b)$ is a finite sum, we apply linearity and Leibniz rule of d to get

$$f'(b) = d(c_0) + bd(c_1) + c_1 d(b) + \cdots + b^n d(c_n) + c_n d(b^n)$$

Since each $c_0, \ldots, c_n \in K$, we obtain $f'(b) = c_1 d(b) + \cdots + c_n \cdot nb^{n-1} d(b)$. Thus factoring out $d(b)$ in the sum, we obtain precisely the standard derivative in calculus, and that $d(f(b)) = f'(b)d(b)$ □

We are ready for the main proposition of the subsection.

Proposition 3.1.2. *Let K be a field and L/K a finite field extension. Then L/K is separable if and only if $\Omega^1_{L/K} = 0$.*

Proof. Suppose that L/K is separable. Suppose that $b \in L$ has minimal polynomial $f \in K[x]$. f is separable since L/K is separable. By 3.1.1, we have that $d(f(b)) = f'(b)d(b)$. But the fact that f is separable implies that $f'(b) \neq 0$. At the same time we have $f(b) = 0$ since f is the minimal polynomial of b. This implies that $d(f(b)) = 0$ in $\Omega^1_{L/K} = 0$. Since L is a field, and $f'(b) \neq 0$, we must have $d(b) = 0$ for all $b \in L$. This means that $\Omega^1_{L/K} = 0$.

If L/K is inseparable, then there exists an intermediate field E such that L/E is a simple inseparable extension. Since L/K is finite, L/E is finite and thus is algebraic which means that there exists some polynomial $p \in E[t]$ for which $L = \frac{E[t]}{(p(t))}$. In this case, we have already seen that

$$\Omega^1_{L/E} \cong \frac{Ldt}{(p'(t)dt)} \cong \frac{L}{(p'(t))}$$

Since $p'(t) = 0$, we have that $\Omega^1_{L/E} \cong L \neq 0$. By the first exact sequence 2.3.1, we have that $\Omega^1_{L/K}$ maps surjectively onto $\Omega^1_{L/E} \neq 0$ which proves that $\Omega^1_{L/K}$ is non-zero.
Materials referenced: [Per15], [Liu06] □

This gives a very nice characterization of separability. Readers can find more in [Har77] and [Liu06]. To extend this equivalence under the assumption that L/K is algebraic instead of finite, one can show that Ω^1 preserves colimits in the sense in [Eis07]. Namely that the functor $F : \text{Algebra}_R \to \text{Mod}_T$ from the category of R-algebra to the category of T-modules where T is a colimit of a diagram in the category of R-algerba preserves colimits. Then observe that an algebraic extension is the colimit of the finite subextensions.

Analogous to the above result, there is a similar proposition for $\text{Der}_K(L)$ for when L/K is algebraic and separable. This is given by [Mor96].

Proposition 3.1.3. *Let L/K be an algebraic field extension that is separable. Then $\text{Der}_K(L) = 0$.*

Proof. Suppose that $D \in \text{Der}_K(L)$. If $a \in L$, let p be the minimal polynomial of a. Then

$$0 = D(p(a)) = p'(a)D(a)$$

by 3.1.1. Since p is separable over K, $p'(a) \neq 0$. Thus $D(a) = 0$ and so we are done.
Materials referenced: [Mor96] □

This proposition will be of use at 4.1.7.

3.2 Detecting Smoothness in Varieties

In manifolds, the the cotangent bundle is a vector bundle of cotangent spaces. Let $m_p = \{f \in \mathbb{C}[V] | f(p) = 0\}$ for a variety V, we have that m_p/m_p^2 is the cotangent space of V from [Sha12]. Motivated by the relation between the cotangent bundles and the cotangent spaces of a manifold, we attempt to recover the cotangent space of a variety from the module of Kähler Differentials.

Combined with the following theorem, we see that by localization, we can see that we recover the cotangent space, at least in the affine, non-scheme theoretic sense:

Theorem 3.2.1. *Let B be a local ring which contains a field K that is isomorphic to B/m the residue field. Then the map*

$$\delta : \frac{m}{m^2} \to \Omega^1_{B/K} \otimes_B K$$

as given in 2.3.2 is an isomorphism.

Proof. Using the second exact sequence 2.3.2, we have that

$$m/m^2 \xrightarrow{\delta} \Omega^1_{B/K} \otimes_B \frac{B}{m} \longrightarrow \Omega^1_{(B/m)/K} \longrightarrow 0$$

But the third term is just $\Omega^1_{K/K}$ which is clearly just 0. Thus δ is surjective. Using the same tactic as in 2.3.2, all we have to do is to show that $w : \mathrm{Der}_K(B, N) \to \mathrm{Der}_K(m, N)$ given by the restriction of scalars is surjective for all K-modules N.
Let $b \in B$. I claim that b is a unique sum of an element in m and an element in B/m. Suppose that $b = c_1 + m_1 = c_2 + m_2$ for $c_1, c_2 \in K$ and $m_1, m_2 \in m$. Then this implies that $c_1 - c_2 \in m$ is a non-unit. But $c_1 - c_2 \in K$ does not have an inverse if and only if $c_1 - c_2 = 0$ thus $c_1 = c_2$. This leaves $m_1 = m_2$.

I claim that the map is surjective as follows. For $h \in \mathrm{Der}_K(m, N)$, define $k \in \mathrm{Der}_K(B, N)$ by $k(b) = k(c + n) = h(n)$ where $c + n$ is the unique representation of b using $c \in R/m$ and $n \in m$. Since the decomposition $b = c + n$ is unique, the map k is well defined. It is moreover B-linear since h is linear.

For $b_1, b_2 \in B$, we have that

$$\begin{aligned} d(b_1 b_2) &= h(c_1 m_2 + c_2 m_1 + m_1 m_2) && (\text{Write } b_i = c_i + m_i \text{ where } c_i \in B/m \text{ and } k_i \in m) \\ &= c_1 h(m_2) + c_2 h(m_1) + h(m_1 m_2) && = c_1 h(b_2) + c_2 h(b_1) + h(0) \\ &= c_1 h(b_2) + c_2 h(b_1) \end{aligned}$$

and

$$\begin{aligned} b_1 d(b_2) + b_2 d(b_1) &= (c_1 + m_1) h(m_2) + (c_2 + m_2) h(m_1) \\ &= c_1 h(b_2) + c_2 h(b_1) \end{aligned}$$

where the second equality follows from the fact that $h(u) = 0$ for $u \in m$. Thus d is a derivation.

We can conclude that v is surjective so that we are done. Materials Referenced: [Har77], [Pro11] □

We are almost ready in recovering the cotangent space. By considering the localization of a coordinate ring $\mathbb{C}[V]$ with a maximal ideal m_p corresponding to points on the variety, we obtain a local ring $\mathbb{C}[V]_{m_p}$ with maximal ideal again m_p. Then the cotangent space $\frac{m_p}{m_p^2}$ as seen in [Sha12], is isomorphic to $\Omega^1_{\mathbb{C}[V]_{m_p}/\mathbb{C}} \otimes_{\mathbb{C}[V]_{m_p}} \mathbb{C}$ by the above theorem. Therefore what remains is to compute the module of Kähler differentials for the localization of a coordinate ring.

Fortunately localization commutes with the construction of the module of Kähler differentials:

Proposition 3.2.2. *Let B be an algebra over A. Let S be a multiplicative subset of B. Then*

$$S^{-1}\Omega^1_{B/A} \cong \Omega^1_{S^{-1}B/A}$$

Proof. This is done in two steps.
Step 1: $\Omega^1_{S^{-1}B/B} = 0$.

We have that for any $u \in S^{-1}B$, there exists some $s \in S$ such that $su \in B$. Applying the canonical derivation gives

$$\begin{aligned} sd(u) &= d(su) && (s \in S \subset B) \\ &= 0 && (su \in B) \end{aligned}$$

Since $s \in S$ is invertible, we must have $d(u) = 0$. Thus $\Omega^1_{S^{-1}B/B} = 0$.

Step 2: Apply the first exact sequence.
By the first exact sequence 2.3.1 and apply it to $C = S^{-1}B$, we obtain a surjective map

$$\Omega^1_{B/A} \otimes_B S^{-1}B \to \Omega^1_{S^{-1}B/A}$$

which by definition of localization of modules, is equal to

$$S^{-1}\Omega^1_{B/A} \to \Omega^1_{S^{-1}B/A}$$

In order to show injectivity of this map, we show that

$$\mathrm{Hom}_{S^{-1}B}(\Omega^1_{S^{-1}B/A}, N) \to \mathrm{Hom}_{S^{-1}B}(S^{-1}\Omega^1_{B/A}, N)$$

is surjective for any $S^{-1}B$-module N. Now the latter module is isomorphic to $\mathrm{Hom}_B(\Omega^1_{B/A}, N)$ by 7.1.3 Using 2.2.2, this is equivalent to showing surjectivity of the map

$$\mathrm{Der}_A(S^{-1}B, N) \to \mathrm{Der}_A(B, N)$$

But this is precisely the content of 2.1.3. So we are done.
Materials referenced: [Liu06] □

In particular, the localization of of a coordinate ring with the maximal ideal recovers the cotangent space of the variety:

Example 3.2.3. Cuspidal Cubic: Part 2 Let us compute the dimensions of the cotangent space of the cuspidal cubic at different points.

Recall that the module of Kähler differentials of the cuspidal cubic is given by

$$\Omega^1_{\mathbb{C}[V]/\mathbb{C}} \cong \frac{\mathbb{C}[V]dx \oplus \mathbb{C}[V]dy}{(-3x^2dx, 2ydy)}$$

Write $m_p = (x - p_1, y - p_2)$ the maximal ideal corresponding to the point $(p_1, p_2) \in V$ by Nullstellensatz. To consider individual cotangent spaces of the variety, we need to first localize the module of Kähler differentials: $\left(\Omega^1_{\mathbb{C}[V]/\mathbb{C}}\right)_{m_p}$.

Notice that for $(p_1, p_2) \neq (0, 0)$, m_p does not contain the elements x and y. This means that x and y are invertible in the localization. Thus within this localization, we can write the relation $-3x^2dx + 2ydy = 0$ as $dy = \frac{3x^2}{2y}dx$. This kills of one of the generators in $\mathbb{C}[V]dx \oplus \mathbb{C}[V]dy$ since we can now express the generator dy with the generator dx. And so we are left with

$$\left(\Omega^1_{\mathbb{C}[V]/\mathbb{C}}\right)_{m_p} \cong \mathbb{C}[V]_{m_p}dx$$

Clearly this is a free $\mathbb{C}[V]_{m_p}$-module of rank 1. Using 3.2.1, we see that

$$\begin{aligned} \frac{m_p}{m_p^2} &\cong \left(\Omega^1_{\mathbb{C}[V]_{m_p}/\mathbb{C}}\right) \otimes_{\mathbb{C}[V]_{m_p}} \frac{\mathbb{C}[V]_{m_p}}{m_p} && \text{(Theorem 3.2.1)} \\ &\cong \left(\Omega^1_{\mathbb{C}[V]/\mathbb{C}}\right)_{m_p} \otimes_{\mathbb{C}[V]_{m_p}} \frac{\mathbb{C}[V]_{m_p}}{m_p} && \text{(Commtues with localization 3.2.2)} \\ &\cong \mathbb{C}[V]_{m_p}dx \otimes_{\mathbb{C}[V]_{m_p}} \frac{\mathbb{C}[V]_{m_p}}{m_p} \\ &\cong \frac{\mathbb{C}[V]_{m_p}}{m_p}dx \\ &\cong \mathbb{C}dx && \text{(Residue field)} \end{aligned}$$

which shows that $\frac{m_p}{m_p^2}$ is a 1-dimensional vector space over $\mathbb{C}$.

However when $(p_1, p_2) = (0, 0)$, things are different. Since localization commutes with quotients (by 3.2.2) and the module of Kähler differentials, we obtain

$$\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}} \cong \frac{\mathbb{C}[V]_{(x,y)}dx \oplus \mathbb{C}[V]_{(x,y)}dy}{((-3x^2)dx \oplus (2y)dy)}$$

Now we claim that there is a surjection $\left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) \to \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)}dx \oplus \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)}dy$ with kernel precisely

$$\left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) x \oplus \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) y$$

In particular, it sends the basis elements $dx \mapsto dx$ and $dy \mapsto dy$.

For surjectivity:
Any element in the codomain is of the form $(k_1 + (x, y))dx \oplus (k_2 + (x, y))dy$ for $k_1, k_2 \in \mathbb{C}$. Then by considering the element $k_1dx \oplus k_2dy \in \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right)$, we see that it precisely maps to $(k_1dx \oplus k_2dy) = (k_1 + (x, y))dx \oplus (k_2 + (x, y))dy$.

The kernel:
We know that $f + (x, y) = (x, y)$ if and only if $f \in (x, y)$. Then $fdx \oplus gdy \in \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right)$ is mapped to $0dx \oplus 0dy$ if and only if $f, g \in (x, y)$. This means that we can rewrite f and g into $f = xf_1 + yf_2$ and $g = xg_1 + yg_2$ so that

$$fdx \oplus gdy = x(f_1dx \oplus g_1dy) + y(f_2dx \oplus g_2dy) \in \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) x \oplus \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) y$$

Together with 3.2.1 and writing $m = (x, y)$, we can conclude that

$$\begin{aligned}
\frac{m}{m^2} &\cong \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) \otimes_{\mathbb{C}[V]_{(x,y)}} \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)} && \text{(Thoerem 3.2.1)}\\
&\cong \frac{\left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right)}{\left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) x + \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) y} && \text{(Proposition 7.1.4)}\\
&\cong \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)}dx \oplus \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)}dy && \text{(The isomorphism we just proved)}\\
&\cong \mathbb{C}dx \oplus \mathbb{C}dy && \text{(Residue field)}
\end{aligned}$$

which shows that $\frac{m}{m^2}$ is a vector space of dimension 2 over $\mathbb{C}$.
Materials Referenced: [Vak22]

While intuitively we know that the ellipse does not have singularities, we still have to be careful of the fact that there are point where the tangent space is a vertical line or a horizontal line.

Example 3.2.4. Ellipse: Part 2 Recall that the module of Kähler differentials for the ellipse $4x^2 + 9y^2 = 36$ is given by

$$\Omega^1_{\mathbb{C}[W]/\mathbb{C}} \cong \frac{\mathbb{C}[W]dx \oplus \mathbb{C}[W]dy}{(8xdx, 18ydy)}$$

Write $m_p = (x - p_1, y - p_2)$ for the maximal ideal corresponding to a point (p_1, p_2) on the ellipse. In a similar fashion as above, we consider the localization

$$\left(\Omega^1_{\mathbb{C}[W]/\mathbb{C}}\right)_{m_p}$$

There are three cases to consider:

Case 1: $p_1, p_2 \neq 0$. Then x and y are invertible in the localization $\left(\Omega^1_{\mathbb{C}[W]/\mathbb{C}}\right)_{m_p}$ since $x, y \notin m_p$. Within the localization, we can now write the relation $8xdx + 18ydy = 0$ as $dy = -\frac{4x}{9y}dx$ thanks to y being invertible. Then

$$\begin{aligned}\left(\Omega^1_{\mathbb{C}[W]/\mathbb{C}}\right)_{m_p} &\cong \left(\frac{\mathbb{C}[W]dx \oplus \mathbb{C}[W]dy}{(8xdx, 18ydy)}\right)_{m_p} \\ &\cong \mathbb{C}[W]_{m_p}dx \oplus \mathbb{C}[W]_{m_p}\left(-\frac{4x}{9y}dx\right) \\ &\cong \mathbb{C}[W]_{m_p}dx \qquad\qquad (4x/9y \in \mathbb{C}[W]_{m_p})\end{aligned}$$

which is free of rank 1.

Case 2: $p_1 = 0$ and so $p_2 = \pm 2$.
Unfortunately $m_p = (x, y - p_2)$ means that x is no longer invertible in the localization, but we can still invert y since $y \notin m_p$. So we write the relation as $dy = \frac{-4x}{9y}dx$ to get $\left(\Omega^1_{\mathbb{C}[W]/\mathbb{C}}\right)_{m_p} \cong \mathbb{C}[W]_{m_p}dx$ which is again, free of rank 1.

Case 3: $p_2 = 0$ and so $p_1 = \pm 3$.
This time $m_p = (x - p_1, y)$ means that y is no longer invertible. However the way to go around this is to instead write dx in terms of dy. Since x is invertible in the localization this time, we have $dx = -\frac{9y}{4x}dy$. A similar argument shows that $\left(\Omega^1_{\mathbb{C}[W]/\mathbb{C}}\right)_{m_p} \cong \mathbb{C}[W]_{m_p}dy$ which is again free of rank 1.

We can conclude that for any point (p_1, p_2) on the variety, $\Omega^1_{\mathbb{C}[W]_{m_p}/\mathbb{C}}$ is free of rank 1. A similar argument as that of the cuspidal cubic shows that the cotangent space has dimension 1 for any point on the ellipse.

Finally we return to the case of the double cone. Its calculations are fairly similar to that of the cuspidal cubic. However since the double cone will have points on it that intersects the xz-plane or yz-plane, we need to apply a similar method as to the one we saw for ellipses.

Example 3.2.5. The Double Cone: Part 2 Recall that the module of Kähler differentials of the double cone $x^2 + y^2 = z^2$ is given by

$$\Omega^1_{\mathbb{C}[U]/\mathbb{C}} \cong \frac{\mathbb{C}[U]dx \oplus \mathbb{C}[U]dy \oplus \mathbb{C}[U]dz}{(2xdx, 2ydy, -2zdz)}$$

Write $m_p = (x - p_1, x - p_2, x - p_3)$ the maximal ideal corresponding to a point $p = (p_1, p_2, p_3)$ on the double cone. Notice that $2x, 2y, 2z \in m_p$ if and only if $p_1 = p_2 = p_3 = 0$. We do a similar case by case analysis as the above examples. There are three cases to consider for the localization $\left(\Omega^1_{\mathbb{C}[U]/\mathbb{C}}\right)_{m_p}$.

Case 1: $(p_1, p_2, p_3) \neq 0$
Then at least one of p_1, p_2, p_3 is non-zero. Correspondingly, at least one of x, y, z is invertible in the localization. To illustrate, suppose that $p_1 \neq 0$. Then x is invertible in the localization and we can write the relation as $dx = \frac{z}{x}dz - \frac{y}{x}dy$. This means that we have written one generator in terms of the other two, which means that now

$$\left(\Omega^1_{\mathbb{C}[U]/\mathbb{C}}\right)_{m_p} \cong \mathbb{C}[U]_{m_p}dy \oplus \mathbb{C}[U]_{m_p}dz$$

which shows that the module of Kähler differentials is free of rank 2. Using 3.2.1 we have that

$$\begin{aligned}\frac{m_p}{m_p^2} &\cong \Omega^1_{\mathbb{C}[U]_{m_p}/\mathbb{C}} \otimes_{\mathbb{C}[U]_{m_p}} \mathbb{C} \\ &\cong \left(\mathbb{C}[U]_{m_p}dy \oplus \mathbb{C}[U]_{m_p}dz\right) \otimes_{\mathbb{C}[U]_{m_p}} \mathbb{C} \\ &\cong \left(\mathbb{C}[U]_{m_p}dy \otimes_{\mathbb{C}[U]_{m_p}} \mathbb{C}\right) \oplus \left(\mathbb{C}[U]_{m_p}dz \otimes_{\mathbb{C}[U]_{m_p}} \mathbb{C}\right) \\ &\cong \mathbb{C}dy \oplus \mathbb{C}dz\end{aligned}$$

which shows that m_p/m_p^2 has dimension 2 as a $\mathbb{C}$-vector space. The case is similar for when $y \neq 0$ and $z \neq 0$.

Case 2: $(p_1, p_2, p_3) = 0$.
Since localization commutes with quotients by 3.2.2, we have

$$\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}} \cong \frac{\mathbb{C}[U]_{(x,y,z)}dx \oplus \mathbb{C}[U]_{(x,y,z)} \oplus \mathbb{C}[U]_{(x,y,z)}dz}{(2xdx \oplus 2ydy \oplus -2zdz)}$$

Now we claim that there is a surjection from this module to

$$\frac{\mathbb{C}[U]_{(x,y,z)}}{(x,y,z)}dx \oplus \frac{\mathbb{C}[U]_{(x,y,z)}}{(x,y,z)}dy \oplus \frac{\mathbb{C}[U]_{(x,y,z)}}{(x,y,z)}dz$$

that sends $dx \mapsto dx$, $dy \mapsto dy$ and $dz \mapsto dz$.

For surjectivity:
Any element in the codomain is of the form $(k_1 + (x, y, z))dx \oplus (k_2 + (x, y, z))dy \oplus (k_3 + (x, y, z))dz$ for $k_1, k_2, k_3 \in \mathbb{C}$. Then by considering the element $k_1 dx \oplus k_2 dy \oplus k_3 dz \in \Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}$, we see that it precisely maps to $k_1 dx \oplus k_2 dy + k_3 dz = (k_1 + (x, y, z))dx \oplus (k_2 + (x, y, z))dy + (k_3 + (x, y, z))dz$.

The kernel:
We know that $v + (x, y, z) = (x, y, z)$ if and only if $v \in (x, y, z)$. Then $fdx \oplus gdy \oplus hdz$ in the domain is mapped to $0dx \oplus 0dy \oplus 0dz$ if and only if $f, g, h \in (x, y, z)$. This means that we can rewrite the three functions as

$$\begin{cases} f &= xf_1 + yf_2 + zf_3 \\ g &= xg_1 + yg_2 + zg_3 \\ h &= xh_1 + yh_2 + zh_3 \end{cases}$$

so that

$$\begin{aligned} fdx \oplus gdy \oplus hdz &= x(f_1dx \oplus g_1dy \oplus h_1dz) + y(f_2dx \oplus g_2dy \oplus h_2dz) + z(f_3dx \oplus g_3dy \oplus h_3dz) \\ &\in \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) x \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) y \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) z \end{aligned}$$

and that $\left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) x \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) y \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) z$ is the kernel of this map.

Now we have an isomorphism

$$\frac{\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}}{\left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) x \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) y \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) z} \cong \bigoplus_{i=1}^{3} \frac{\mathbb{C}[U]_{(x,y,z)}}{(x,y,z)} dx_i$$

(where we write x_1 as x, x_2 as y and x_3 as z for simplicity).

Together with 3.2.1, and writing $m = (x, y, z)$, we can conclude that

$$\begin{aligned} \frac{m}{m^2} &\cong \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) \otimes_{\mathbb{C}[U]_{(x,y,z)}} \frac{\mathbb{C}[U]_{(x,y,z)}}{(x,y,z)} && \text{(Theorem 3.2.1)} \\ &\cong \frac{\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}}{\left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) x \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) y \oplus \left(\Omega^1_{\mathbb{C}[U]_{(x,y,z)}/\mathbb{C}}\right) z} && \text{(Proposition 7.1.4)} \\ &\cong \bigoplus_{i=1}^{3} \frac{\mathbb{C}[U]_{(x,y,z)}}{(x,y,z)} dx_i \\ &\cong \mathbb{C}dx \oplus \mathbb{C}dy \oplus \mathbb{C}dz \end{aligned}$$

which shows that the cotangent space at the origin has dimension 3.

This matches nicely with the geometric picture of the double cone. Every non-zero point on the double cone has cotangent space of dimension 2.

Recall that a point on the variety is singular if the dimension of the cotangent space is strictly greater than the dimension of the variety. The module of Kähler Differentials gives us a way to find out which points are the singularities of the varieties by analyzing the Jacobian of the equations defining the variety (Indeed the Jacobian is encoded in the module of Kähler Differentials as the quotient relation).

4 Relation of the Module of Kähler Differentials with Manifolds

The previous section showed that given the module of Kähler differentials over a coordinate ring, we can determine the dimension of the cotangent space of the corresponding variety, at different points. In the context of manifold theory, smooth 1-forms are smooth sections of the cotangent bundle, while we can recover the cotangent space of point of the manifold from the cotangent bundle. This motivates the following section. In particular, we compare the two constructions and would like to find out how similar are the two.

4.1 The Global Case: Vector Fields and Smooth 1-Forms

We have encountered in MA3H5 Manifolds the definition of vector fields and 1-forms. It has been done in a very geometric way by visualizing a smooth assignment of tangent / cotangent vectors for each point on the manifold. There is also a very algebraic way of describing the tangents that reveals more structure on these vectors.

The below definition is given in [Tu10] P.136.

Definition 4.1.1. Smooth Vector Field Let M be a smooth manifold. A smooth vector field is a smooth section $X : M \to TM$ from M to the tangent bundle TM. The set of all smooth vector fields is denoted by $\mathfrak{X}(M)$.

We will not prove that $\mathfrak{X}(M)$ has the structure of a vector space here and we will take this fact for granted. Interested readers can refer to [Tu10].

If we take the $\mathbb{R}$-algebra $C^\infty(M)$ as a module over itself, it makes sense to talk about the set of all derivations $\text{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$ from the $\mathbb{R}$-algebra to itself. Let us denote this by the shorthand notation $\text{Der}_{\mathbb{R}}(C^\infty(M))$. Note that here we are talking about derivations of the algebra, not derivations at a point p of the manifold, as noted in [Tu10] P.17.

[Tu10] gave an isomorphism between the vector spaces $\mathfrak{X}(M)$ of all smooth vector fields and $\text{Der}_{\mathbb{R}}(C^\infty(M))$ in the case $M = \mathbb{R}^n$. It also gave out steps in how one would go to prove this for general smooth manifolds.

Proposition 4.1.2. *Let M be a smooth manifold. The map*

$$\phi : \mathfrak{X}(M) \to Der_{\mathbb{R}}(C^\infty(M))$$

that sends $X \mapsto (f \mapsto Xf)$ defines an isomorphism of vector spaces.

Proof.
Step 0: $\phi(X)$ is a derivation.
By definition we have that $\phi(X)(f) = Xf$. We want to show that $Xf \in C^\infty(M)$. Let $(U, \phi = (x^1, \ldots, x^n))$ be a chart on M. Then X can be written as $\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ for some C^∞ function a^i in the chart. It follows that $Xf = \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i}$ is C^∞. Since M can be covered by such charts, we have that Xf is C^∞ on M. $\phi(X)$ is $\mathbb{R}$-linear and satisfies the Leibniz rule since the partial derivatives $\frac{\partial}{\partial x^i}$ satisfies them on a local expression of X. This means that X also satisfies them.

Step 1: ϕ is a $C^\infty(M)$-linear map.
Let $X, Y \in \mathfrak{X}(M)$. For any $f \in C^\infty(M)$ and $p \in M$, we have $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$ since T_pM is a vector space. This means that as p varies, we have $(X + Y)(f) = Xf + Yf$. $X + Y$ is smooth since smooth sections sum to smooth sections.

Now let $g \in C^\infty(M)$. We want to show that $\phi(gX)(f) = g\phi(X)(f)$ for any $f \in C^\infty(M)$. But we have on local coordinates:

$$gX(f) = g\sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} = \sum_{i=1}^n g a^i \frac{\partial f}{\partial x^i} = (gX)(f)$$

Step 2: ϕ is injective.
Suppose that $X \in \mathfrak{X}(M)$ is such that $Xf = 0$ for any f. Let $(U, \phi = (x^1, \dots, x^n))$ be a chart. On the charts, Xf can be expressed as $Xf = \sum_{i=1}^{n} c^i \frac{\partial f}{\partial x^i}$ locally. Choose f such that $\frac{\partial f}{\partial x^j}$ is zero for all j other than 1. Then we have that $0 = Xf = c^1 \frac{\partial f}{\partial x^1}$ which shows that c^1 must be zero. We can do the same thing for $c^2, \dots, c^n$ to show that locally, $c^1 = \cdots = c^n = 0$. Since M can be covered by such charts, we have that $X = 0$.

Step 3: Define a new map D_p and show that it is well defined.
Let $D \in \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$. Define $D_p : C^\infty_{M,p} \to C^\infty_{M_p}$ by $D_p([f]) = [D\overline{f}]$ where $\overline{f}$ is any global extension of f (this is possible by partition of unity). We want to show that for different choices $g, h \in [f]$, $[D\overline{g}] = [D\overline{h}]$. Now if $g, h \in [f]$, then there exists some open set $U \subseteq M$ such that $g|_U = h|_U$. Then this means that $D\overline{g}|_U = D\tilde{h}|_U$ and thus $D\overline{g}$ and $D\overline{h}$ lie the same equivalence class: $[D\overline{g}] = [D\overline{h}]$.

Step 4: D_p is a derivation at a point p.
I want to show that $D_p \in \mathrm{Der}_{\mathbb{R}}(C^\infty_{M,p})$. This means that we need to check $\mathbb{R}$-linearity and that it satisfies the Leibniz rule.

- $\mathbb{R}$-linearity: Let $a \in \mathbb{R}$. I claim that $a[f] = [af]$. Let $g \in [f]$. Then $g|_U = f|_U$ for some open set U of M. This is true if and only if $ag|_U = af|_U$ thus $ag \in [af]$. Then we have

$$\begin{aligned}
D_p(a[f]) &= D_p([af]) \\
&= [D\overline{af}] \\
&= [D(a\overline{f})] && \text{(Extension is linear)} \\
&= [aD(\overline{f})] && (D \text{ is } \mathbb{R}\text{-linear}) \\
&= a[D\overline{f}] \\
&= aD_p([f])
\end{aligned}$$

- Leibniz rule: Let $[f], [g] \in C^\infty_{M,p}$. Then we have

$$\begin{aligned}
D_p([f] \cdot [g]) &= D_p([fg]) \\
&= [D\overline{fg}] \\
&= [D\left(\overline{f}\overline{g}\right)] \\
&= [\overline{f}D\overline{g} + \overline{g}D\overline{f}] \\
&= [\overline{f}][D\overline{g}] + [\overline{g}][D\overline{f}] \\
&= [f]D_p([g]) + [g]D_p([f])
\end{aligned}$$

This shows that D_p is a derivation at a point p.

Step 5: ϕ is surjective and thus ϕ is an isomorphism of $C^\infty(M)$-modules.
In step 4, to every $D \in \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$ we associated a tangent vector D_p. Let $X : M \to TM$ be defined as $X(p) = D_p$. It remains to show X is a smooth vector field and that $\phi(X) = D$.

Let $f \in C^\infty(M)$. We claim that $D_p([f])$ glue together into Df which is a smooth function. Clearly, Df is a smooth function on M that lies in each $[Df] \in C^\infty_{M,p}$. This Df is also unique: Suppose that g is a globally smooth function that also lies in each $[Df] \in C^\infty_{M,p}$. Then there exists some neighbourhood U_p of p such that $Df|_U = g|_U$. But all the U_p cover M. Thus $Df = g$. It is also clear that $\phi(X)(f) = Df$ for every $f \in C^\infty(M)$. Thus $\phi(X) = D$. Materials referenced: [Tu10] □

This is an unfortunate mess of notation! The X on the domain of the map ϕ is a function $M \to TM$ while the map $f \mapsto Xf$ which is typically also indicated by X, sends a $C^\infty(M)$ function to a $C^\infty(M)$ function.

Finally, let us also recall the definition of smooth 1-forms on M. The definition below is also given in [Tu10] P.193

Definition 4.1.3. Smooth 1-Forms Let M be a smooth manifold. A smooth 1-form on M is a smooth section $\omega : M \to T^*M$ from M to the cotangent bundle. The set of all smooth 1-forms is denoted by $\Omega^1(M)$.

Similar to $\mathfrak{X}(M)$, there is a vector space structure on $\Omega^1(M)$ which we will not prove and take it for granted again. Once again, readers can refer to [Tu10].

Considering the similarities between smooth vector fields and smooth 1-forms in their definition, we expect them to be somewhat related. Indeed we have the following proposition.

Proposition 4.1.4. *Let M be a smooth manifold. On local coordinates, write $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x_i}$ for $X \in Der_{\mathbb{R}}(C^\infty(M))$ and $\omega = \sum_{i=1}^n b^i dx_i$ for $\omega \in \Omega^1(M)$. Define a pairing $\psi : Der_{\mathbb{R}}(C^\infty(M)) \times \Omega^1(M) \to C^\infty(M)$ by*

$$(X, \omega) \mapsto \omega(X)\left(p \mapsto \sum_{i=1}^n b^i(p)a^i(p)\right)$$

locally. Then this is a dual pairing and hence induces an isomorphism

$$\Omega^1(M) \cong \mathrm{Hom}_{C^\infty(M)}(Der_{\mathbb{R}}(C^\infty(M)), C^\infty(M))$$

Proof. Firstly, note that this definition makes sense. On local coordinates, write $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x_i}$ for $X \in \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$ and $\omega = \sum_{i=1}^n b^i dx_i$ for $\omega \in \Omega^1(M)$. Then locally we have that

$$\begin{aligned} \omega(X) &= \sum_{k=1}^n b^k dx_k \left(\sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\right) \\ &= \sum_{k=1}^n b^k a^k & \left(dx_k\left(\tfrac{\partial}{\partial x^j}\right) = \delta_{i,j}\right) \end{aligned}$$

Since each b^i and a^i are smooth, $\omega(X)$ is also smooth locally and hence $\omega(X)$ is smooth globally. Now we show that this is a dual pairing. Suppose first that $\psi(X, \omega) = 0$ for all $X \in \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$. Fix $k \in \{1, \dots, n\}$. Choose $X \in \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$ such that $a^j = 0$ for any $j \neq k$ and on any chart of M. Then $\psi(X, \omega) = 0$ implies $b^k a^k = 0$ for $a^k \neq 0$. Thus $b^k = 0$. Repeating this argument for each $k \in \{1, \dots, n\}$ shows that $b^1 = \dots = b^n = 0$ on any chart of M and thus $\omega = 0$. A similar method shows that if $\psi(X, \omega) = 0$ for all $\omega \in \Omega^1(M)$, then $X = 0$. We conclude that ψ is a dual pairing and hence induces the required isomorphism. □

Now that we have the definitions at hand, we turn back to its relation with the module of Kähler differentials. In particular, how is the module of Kähler differentials related to the smooth 1-forms? Recall that for each manifold, there is an $\mathbb{R}$-algebra of smooth functions on M, given by

$$C^\infty(M) = \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}$$

We have the following result:

Proposition 4.1.5. *Let M be a smooth manifold. Then we have an isomorphism of modules*

$$\left(\Omega^1_{C^\infty(M)/\mathbb{R}}\right)^{**} \cong \Omega^1(M)$$

Proof. Applying $C^\infty(M)$ to lemma 2.2.2, we obtain the expression

$$\mathrm{Hom}_{\mathbb{R}}\left(\Omega^1_{C^\infty(M)/\mathbb{R}}, C^\infty(M)\right) \cong \mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M)) = \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$$

This shows that $\left(\Omega^1_{C^\infty(M)/\mathbb{R}}\right)^* = \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$.
Now on one hand, taking the $C^\infty(M)$-module dual of $\mathrm{Der}_{\mathbb{R}}(C^\infty(M))$ again results in the double dual $(\Omega^1_{C^\infty(M)/\mathbb{R}})^{**}$. On the other hand, by definition, we know that $\Omega^1(M)$, the space of smooth 1-forms, is the $C^\infty(M)$-module dual of $\mathrm{Der}_{\mathbb{R}}(C^\infty(M))$. This means that we have

$$\left(\Omega^1_{C^\infty(M)/\mathbb{R}}\right)^{**} \cong \Omega^1(M)$$

Thus we are done. □

Unfortunately, general modules do not have the nice property that double duals are canonically isomorphic to the module itself. So we cannot conclude that $\Omega^1_{C^\infty(M)/\mathbb{R}}$ and $\Omega^1(M)$ is "the same" up to isomorphism. The best that we can do is is a canonical homomorphism $B \to B^{**}$ for any A-module B. [Bou73] P.239 has a brief section on double duals of a module.

In terms of manifolds, we can prove the existence of the canonical homomorphism easily.

Lemma 4.1.6. *Let M be a smooth manifold. Then the exterior derivative $d : C^\infty(M) \to \Omega^1(M)$ induces a unique $C^\infty(M)$-module homomorphism*

$$\phi : \Omega^1_{C^\infty(M)/\mathbb{R}} \to \Omega^1(M)$$

given by the universal property of the module of Kähler differentials.

Proof. We know that there is the exterior derivative $d : C^\infty(M) \to \Omega^1(M)$ sending a smooth function on M to its 1-form. This map is an $\mathbb{R}$-linear map since scalar multiplication of $\mathbb{R}$ can be factored outside. The exterior derivative also satisfies the Leibniz rule. Thus, by the universal property of the module of Kähler differentials, the required map exists and is unique. $\square$

One way to think of the failure of bijectivity is to consider what happens to analytic functions. Take M to be the real line $\mathbb{R}$ for simplicity. The function e^x, under the exterior derivative gets sent to $e^x dx$. However, considering the construction of the module of Kähler differentials using the quotient of the free module, we see that we can only perform the Leibniz rule and linearity rule only a finite amount of times, whereas e^x is a Taylor polynomial of countable many terms.

Notice that since the exterior derivative is $\mathbb{R}$-linear, it is an $\mathbb{R}$-derivation and thus $\Omega^1(C^\infty(\mathbb{R}))$ factors through $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$ by the universal property. In $\Omega^1(C^\infty(\mathbb{R}))$, $d_{\text{ext}}(e^x) = e^x d_{\text{ext}}(x)$. The map $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}} \to \Omega^1(C^\infty(\mathbb{R}))$ given by the universal property is defined by $d(f) \mapsto d_{\text{ext}}(f)$. This means that $d(e^x)$ and $e^x d(x)$ map to the same element in $\Omega^1(C^\infty(\mathbb{R}))$. But whether $d(e^x)$ and $e^x d(x)$ are the same element in $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$ is a question of injectivity of this map.

Below is an idea of how $\Omega^1(\mathbb{R})$ is not isomorphic to $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$ when considering $\mathbb{R}$ as a manifold. The following proof is modified and is based on a Maths Overflow discussion: [hes]

Example 4.1.7. Consider $\mathbb{R}$ as a smooth manifold. Then $\Omega^1(\mathbb{R})$ is not isomorphic to $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$. In particular, for $f(x) = x$ and $g(x) = e^x$, $d(e^x) = e^x d(x)$ in $\Omega^1(\mathbb{R})$ but $d(e^x)$ and $d(x)$ are linearly independent in $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$.

Proof.
Consider the ring $C^\infty(M)$. Let D be a non-principal ultra filter 7.3.1 on $\mathbb{N}$. Define

$$I = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \{n \in \mathbb{N} \mid f(n) = 0\} \in D \right\}$$

We show that I is a maximal ideal. It is an ideal since for $f, g \in I$, then

$$\{n \in \mathbb{N} \mid f(n) + g(n) = 0\} \supseteq \{n \in \mathbb{N} \mid f(n) = 0\} \cap \{n \in \mathbb{N} \mid g(n) = 0\} \in D$$

By property 2 and 3 in definition 7.3.1, we have that $\{n \in \mathbb{N} \mid f(n) + g(n) = 0\} \in D$ so that $f + g \in I$. Let $r \in \mathbb{R}$. Then $\{n \in \mathbb{N} \mid rf(n) = 0\} = \{n \in \mathbb{N} \mid f(n) = 0\} \in D$ when $r \neq 0$. When $r = 0$, we have that $\{n \in \mathbb{N} \mid rf(n) = 0\} = \mathbb{N}$. By property 1 of definition 7.3.1 we have $\mathbb{N} \in D$ so that either way, $rf \in I$.

Consider the coset $f + I$ for $f \notin I$. We want to show that it has an inverse. If $f \notin I$, then $\{n \in \mathbb{N} \mid f(n) \neq 0\} \in D$ by the property of an ultrafilter. We can find $g \in C^\infty(\mathbb{R})$ such that $g(n) = \frac{1}{f(n)}$ for all $n \in \mathbb{N}$ such that $f(n) \neq 0$ (See 7.4.2). Then $\{n \in \mathbb{N} \mid f(n) \neq 0\} \in D$ implies that $\{n \in \mathbb{N} \mid f(n)g(n) = 1\} \in D$. This implies that $fg - 1 \in I$. Thus I is a maximal ideal of $C^\infty(\mathbb{R})$.

Now $K = \frac{C^\infty(\mathbb{R})}{I}$ is a field. $\mathbb{R}$ is embedded as a subfield of K by the field homomorphism defined by $c \mapsto (f(x) = c) + I$. Moreover, $[f(x) = x]$ and $[g(x) = e^x]$ in K are algebraically independent. Indeed, if $p(a, b) = \sum_{i,j} u_{i,j} a^i b^j$ is a polynomial in $\mathbb{R}[x, y]$, we have that $p([x], [e^x]) \in I$ if and only if $\{u \in \mathbb{R} \mid p(u, e^u) = 0\} \in D$. But p is a polynomial and so can only has at most a finite number of solutions. This means that $\{u \in \mathbb{R} \mid p(u, e^u) = 0\}$ is finite. But then this set cannot be in D because

Filter contains finite sets if and only if it is principal by 7.3.6). Thus $p([x], [e^x]) \notin I$ and hence they are algebraically independent. Choose a transcendence basis $S = \{[f(x) = x], [g(x) = e^x], z_3, z_4, \dots\}$ for $K/\mathbb{R}$.

We now have the following extension of fields:

$$\mathbb{R} < \mathbb{R}(S) < K$$

Any $\mathbb{R}$-derivation d on $\mathbb{R}(S)$ is uniquely determined its values on S. This fact is given in [ZS75] Ch2.17 Example 4. In particular we can choose the values that $d([x])$ and $d([e^x])$ take such that they are linearly independent. Since S is a transcendence basis, $K/\mathbb{R}(S)$ is an algebraic extension. It is more over separable since K is a field extension of $\mathbb{R}$ which has characteristic 0. By 3.1.3, $\mathrm{Der}_{\mathbb{R}(S)}(K) = 0$. This means that any $\mathbb{R}$-derivation on $\mathbb{R}(S)$ can be extended uniquely to K. Indeed, if d_1, d_2 are extensions of an $\mathbb{R}$-derivation d over $\mathbb{R}(S)$, then $d_1 - d_2$ is an $\mathbb{R}(S)$-derivation so that

$$d_1 - d_2 \in \mathrm{Der}_{\mathbb{R}(S)}(K) = 0$$

which implies that $d_1 = d_2$.

Now let $d : \mathbb{R}(S) \to \mathbb{R}(S)$ be an $\mathbb{R}$-derivation. By the above digression it can be extended uniquely to an $\mathbb{R}$-derivation $d : K \to K$. Denote $p : C^\infty(\mathbb{R}) \to \frac{C^\infty(\mathbb{R})}{I} = K$ the $C^\infty(\mathbb{R})$-linear projection map and in particular is an $\mathbb{R}$-linear map. Now consider the map $D = d \circ p$. Since d and p are $\mathbb{R}$-linear, D is $\mathbb{R}$-linear. Also, since p is $C^\infty(\mathbb{R})$-linear and d satisfies the Leibniz rule, we conclude that D also satisfies the Leibniz rule. So we now have an $\mathbb{R}$-derivation $D : C^\infty(\mathbb{R}) \to K$. By the universal property of the module of Kähler differentials, we obtain a factorization

$$\begin{tikzcd} C^\infty(\mathbb{R}) \arrow[r, "d^u"] \arrow[dr, "D"'] & \Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}} \arrow[d, dashed, "\exists! q"] \\ & K \end{tikzcd}$$

where d^u denotes the universal derivation associated with $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$.

By the above digression, $D(x)$ and $D(e^x)$ are linearly independent in K. But this means that $d^u(x)$ and $d^u(e^x)$ are linearly independent in $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$. Because otherwise, if they are linearly dependent, then $q(d^u(x)) = D(x)$ and $q(d^u(e^x)) = D(e^x)$ would have linear relations, a contradiction. $\square$

In terms of the global constructs on a manifolds, we have the following diagram

$$
\begin{tikzcd}
& C^\infty(M) \arrow[dl, dashed, "\text{Construct the module}"'] \arrow[dr, "d"] \arrow[dddr, bend left, "\text{Exterior Derivative}"] & \\
\mathrm{Der}(C^\infty(M)) \arrow[drr, "\text{Dualize again}"] \arrow[dddrr, "\text{Dual by Definition}"'] & & \Omega^1_{C^\infty(M)/\mathbb{R}} \arrow[ll, "\text{Dualizes to}"'] \arrow[d, "\text{Canonical Hom}"] \\
& & \left(\Omega^1_{C^\infty(M)/\mathbb{R}}\right)^{**} \arrow[d, "\cong"] \\
& & \Omega^1(M)
\end{tikzcd}
$$

where $\mathrm{Der}(C^\infty(M))$ are the smooth vector fields and $\Omega^1(M)$ are the smooth 1-forms. By [Bou73], the isomorphism does not occur frequently. One such criteria for isomorphism is for $\Omega^1_{C^\infty(M)/\mathbb{R}}$ to be a finitely generated projective module.

As a final note, by considering $C^\infty(-)$ as a sheaf of algebras on a manifold M, we have a completely analogous result, such as

$$\Omega^1(U) \cong \mathrm{Hom}_{C^\infty(U)/\mathbb{R}}(\mathrm{Der}_{\mathbb{R}}(C^\infty(U)), C^\infty(U))$$

This leads to the natural question of whether this generalizes well into the germs of the sheaf. Namely, can we identify similar isomorphisms as above for tangent spaces and cotangent spaces? The following subsection will extend on this.

4.2 The Local Case: Tangent Spaces and Cotangent Spaces

While we have seen the connection between globally smooth 1-forms and the module of Kähler differentials, we have yet to see the connection locally. Analogous to the global constructions where smooth vector fields are equal to $\mathrm{Der}_{\mathbb{R}}(C^\infty(M))$ and smooth 1-forms are equal to $\mathrm{Der}(C^\infty(M))^*$, we can also define tangents and cotangent vectors in a similar fashion. A crucial fact is the following.

Proposition 4.2.1. *Let M be a smooth manifold. Then $\mathcal{C}^\infty_{M,p}$ is a local ring with maximal ideal*

$$m_p = \{f \in \mathcal{C}^\infty_{M,p} \mid f(p) = 0\}$$

Proof. m_p is clearly an ideal since $f, g \in m_p$ means that there is some neighbourhood U_f for which $f(p) = 0$ and some neighbourhood U_g for which $g(p) = 0$. This implies $f(p) + g(p) = 0$ on any open set $W \subset U_f \cap U_g$. Also $r \in \mathcal{C}^\infty_{M,p}$ implies $r(p)f(p) = 0$ in U_f since $f(p) = 0$.
To see that this ideal is maximal, notice that cosets of m_p are exactly of the form $x + m_p = \{f \in \mathcal{C}^\infty_{M,p} | f(p) = x\}$ for $x \in \mathbb{R}$. So we have

$$\frac{C^\infty_{M,p}}{m_p} \cong \mathbb{R}$$

which is a field.
This maximal ideal is clearly unique since it consists precisely of its non-units. □

The standard definition of the tangent space is given in terms of the above local ring and derivations. The following definition is given in both [Tu10] and in MA3H5:

Definition 4.2.2. (Co)Tangent Spaces Let M be a smooth manifold. Let $p \in M$. The tangent space of M at p is

$$T_pM = \text{Der}_{\mathbb{R}}(C^\infty_{M,p}, \mathbb{R})$$

The cotangent space of M at p is the vector space dual of the tangent space, denoted T^*_pM.

Notice that in the definition of $\text{Der}_A(B, M)$ in **??**, we require that M is a B-module. So how is $\mathbb{R}$ a $C^\infty_{M,p}$-module? The answer lies in 4.2.1. It say that $\mathbb{R} \cong \frac{C^\infty_{M,p}}{m_p}$ so that $\mathbb{R}$ can be thought of as the quotient ring of $C^\infty_{M,p}$, which is where the module structure of $C^\infty_{m,p}$ comes from.

Similar to 4.1.5 where there is a relation between vector fields and differential 1-forms (global versions of tangent spaces and cotangent spaces), we can also establish a connection between the (co)tangent space and module of Kähler differentials.

Proposition 4.2.3. *Let M be a smooth manifold and $p \in M$ be a point. Then*

$$T^*_pM \cong \Omega^1_{C^\infty_{M,p}/\mathbb{R}} \otimes_{C^\infty_{M,p}} \mathbb{R}$$

is the cotangent space of M at p.

Proof. Using 3.2.1, we obtain an isomorphism $m_p/m_p^2 \cong \Omega^1_{C^\infty_{M,p}/\mathbb{R}} \otimes_{\mathbb{C}^\infty_{M,p}} \mathbb{R}$ where $m_p = \{f \in C^\infty_{M,p} \mid f(p) = 0\}$. Define a pairing

$$\phi : \frac{m_p}{m_p^2} \times T_pM \to \mathbb{R}$$

by $\phi(f, X_p) = X_pf$. We show that this is a dual pairing. Suppose that $\phi(f, X_p) = 0$ for all $X_p \in T_pM$. By Taylor's theorem (Theorem C.15 in [Lee03]), we have that on a local chart,

$$f(x) = f(p) + \sum_{k=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_p (x_i - p_i) + \sum_{k=1}^{n} u_i(x)(x_i - p_i)$$

where each u_i are C^∞ in the chart and $u_i(p) = 0$. But since $f \in m_p/m_p^2$, this means that $f(p) = 0$. Together with $X_pf = 0$, we are left with f being identified in m_p/m_p^2 as $\sum_{k=1}^n u_i(x)(x_i - p_i)$. But each $u_i(x)$ and $x_i - p_i$ lie in m_p implies that $f \in m_p^2$.

Now suppose that $\phi(f, X_p) = 0$ for all $f \in m_p/m_p^2$. In local coordinates this means that

$$\sum_{k=1}^{n} a_i \frac{\partial f}{\partial x_i}\bigg|_p = 0$$

for each a_i being C^∞, dependent on X. Then in particular, the function $u_i(x) = x_i - p_i$ defined locally on p lies in m_p with only non zero partial derivative being $\frac{\partial u}{\partial x_i}$. Substituting this into the expression, we get $a_i \frac{\partial u}{\partial x_i} = 0$. This leaves us with $a_i = 0$. Repeating the argument for each i, we see that $a_1 = \cdots = a_n = 0$

which means that $X_p = 0$.

The dual pairing then implies that the cotangent space is given by

$$T_p^* M \cong \frac{m_p}{m_p^2}$$

and so we conclude. □

Given a smooth manifold M and its cotangent bundle $p : T^*M \to M$, for any point x on the manifold we can obtain its cotangent space by $p^{-1}(x)$. The above proposition shows that we can use the module of Kähler differentials to recover the cotangent space as well.

5 Conclusion

5.1 What we have done

In the first part, we gave a number of isomorphic constructions of the module of Kähler differentials. We have also seen some of its first results, namely the first and second exact sequences and used them to compute the module of Kähler differentials of coordinate rings.

In the second part, we have seen from 3.2.1 that we can recover the cotangent space from the module of Kähler differentials, showing that it bears similarity with the smooth 1-forms / cotangent bundle on manifolds. We also used the module to find the dimension of the cotangent spaces. There is also a brief discussion on the relation of the module of Kähler differentials and separable field extensions.

However in the last part, we then showed that it is only the double dual that actually resembles the smooth 1-forms in the case of manifolds. Nonetheless, we are able to at least recover the classical cotangent space of a variety using the localization of its coordinate ring into the maximal ideal corresponding to the point. In fact, [Har77] does show that this construction can be made into the relative cotangent sheaf by the tilde construction $(\Omega^1_{X/Y})^\sim$ and thus works well with schemes. There is a brief collection of materials relating to this sheaf from [Har77], [Liu06] and [HMS17].

5.2 Looking Forward

There are many more ways of working with the sheaf of Kähler differentials. In the theory of manifolds, we use the algebra of smooth differential forms together with the exterior derivative to form a cochain complex. This then gives the de Rham cohomology of a smooth manifold. We can also do the same for Kähler differentials. Namely, by constructing the exterior algebra of the module of Kähler differentials and extending the the universal derivation, we also obtain a cochain complex which gives us a cohomology.

Given the wide deployment of scheme theory in algerbaic geometry, one can also turn Kähler differentials into a sheaf. This is done by mimicking the construction in proposition 2.2.4. Interested readers are referred to [Har77] and [Liu06] .

Throughout our journey, we have also established some connection between the module of Kähler differentials and field theory. Separability in fields of characteristic 0 is characterized by the fact that minimal polynomials and its formal derivative is coprime. Intuitively it makes sense for the module of Kähler differentials is related to this notion since they both are related to derivatives. Advanced treatment of the relationship can be found in [Eis07], [ZS75] and [Mat80].

We have omitted the fact that Ω^1 works well between coproducts and coequalizers in the category of algebras over a fixed ring R. [Eis07] proves the two special cases of colimits (coproducts and coequalizers), thus proving that the functor

$$F : \text{Algebra}_R \to \text{Mod}_T$$

where T is the colimit of a diagram in Algebra_R, defined by $S \mapsto T \otimes_S \Omega^1_{S/R}$ and

$$(\varphi : S \to S') \mapsto \left(1 \otimes D\varphi : T \otimes_S (S \otimes_{S'} \Omega^1_{S'/R}) \to T \otimes_S \Omega^1_{S/R}\right)$$

preserves colimits. As suggested in section 3.1, this allows the characterization of separability to be extended from the case of finite extensions to algebraic extensions since algebraic extensions are colimits of its finite subextensions. There are also functorial properties of Ω^1 in which [Eis07] contains.

Let R be a ring and let M be an R-module. Define the Hoschild complex to be the chain complex $C(R, M)$ given as follows.

$$\cdots \longrightarrow M \otimes R^{\otimes n+1} \xrightarrow{d} M \otimes R^{\otimes n} \xrightarrow{d} M \otimes R^{\otimes n-1} \longrightarrow \cdots \longrightarrow M \otimes R \longrightarrow M \longrightarrow 0$$

The map d is defined by $d = \sum_{i=0}^{n}(-1)^i d_i$ where $d_i : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1}$ is given by the following formula.

- If $i = 0$, then $d_0(m \otimes r_1 \otimes \cdots \otimes r_n) = mr_1 \otimes r_2 \otimes \cdots \otimes r_n$
- If $i = n$, then $d_n(m \otimes r_1 \otimes \cdots \otimes r_n) = r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}$
- Otherwise, then $d_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{n-1}$

The cohomology of this cochain complex is called Hochschild cohomology and is denoted by $HH^n(R; M)$. It is a direct generalization of group cohomology. The Hochschild-Kostant-Rosenberg theorem states that when we choose $M = A$ a smooth algebra over a field $R = k$, then we obtain an isomorphism

$$HH^1(R; M) \cong \Omega^1_{A/k}$$

More generally, by wedging the module of Kähler differentials n times with itself, we obtain an isomorphism between the wedge and the nth Hochschild cohomology. This ties in the use of the module in Algebraic Topology and is closely related to the trace map and K-theory. Interested readers are referred to [Lod97].

6 References

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7 Appendix

7.1 Brief Section on Modules

In this section we collect some theorems on modules that will prove itself to be useful later. All rings are assumed to be commutative with $1 \neq 0$.

The first half of this section will consists of theorems related to the set of all R-module homomorphisms $\mathrm{Hom}_R(M, N)$ for M, N R-modules. The second half is dedicated to tensor products and its relation to various constructs of modules. There will also be theorems related to free modules closer to the end.

Theorem 7.1.1. *Let R be a ring and M, N be R-modules. Then the set*

$$\mathrm{Hom}_R(M, N) = \{f : M \to N | f \textit{ is an } R\textit{-module homomorphism}\}$$

is an R-module.

Proof. Let $f, g \in \mathrm{Hom}_R(M, N)$. Define $f+g : M \to N$ by $m \mapsto f(m)+g(m)$. $f+g$ is indeed an R-module homomorphism since

- Addition is preserved: For $m_1, m_2 \in M$,

$$\begin{aligned}(f+g)(m_1+m_2) &= f(m_1+m_2)+g(m_1+m_2)\\ &= f(m_1)+f(m_2)+g(m_1)+g(m_2)\\ &= (f+g)(m_1)+(f+g)(m_2)\end{aligned}$$

- Scalar multiplication is preserved: For $r \in R$ and $m \in M$,

$$\begin{aligned}(f+g)(r \cdot m) &= f(r \cdot m)+g(r \cdot m)\\ &= r \cdot f(m)+r \cdot g(m)\\ &= r \cdot (f+g)(m)\end{aligned}$$

This shows that this operation is closed under $\mathrm{Hom}_R(M, N)$.
This operation also allows $\mathrm{Hom}_R(M, N)$ to be an abelian group since the axioms are satisfied:

- Associativity: Follows from associativity of addition in M.
- Identity: The zero map 0 since $(f+0)(m) = f(m)+0 = f(m)$ for each $m \in M$. Thus $f+0 = f$
- Inverse: The map $m \mapsto -f(m)$ for each $m \in M$ is the inverse of $f : M \to N$. Clearly it is equal to the zero map.
- Abelian: Follows from the fact that M is abelian.

Define an action on $\mathrm{Hom}_R(M, N)$ by $\cdot : R \times \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N)$ where $r \cdot f$ is the function taking $m \in M$ to $r \cdot f(m)$. Associativity clearly follows since N is an R-module. The identity 1 also gives the trivial action. Thus we are done. $\square$

Theorem 7.1.2. *Suppose that A, B, C are R modules. Suppose that $f : A \to B$ and $g : B \to C$ are R-module homomorphisms. Then the following sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if the following sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g_*} \mathrm{Hom}_R(B, N) \xrightarrow{f_*} \mathrm{Hom}_R(A, N)$$

is exact for every R-module N.

Proof. Suppose first that $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact. Clearly g_* is defined as $\phi \mapsto \phi \circ g$ and similarly for f_*. To show that g_* is injective, suppose that $\phi \in \ker(g_*)$. Then $\phi \circ g = 0$ which means that $\mathrm{im}(g) \subseteq \ker(\phi)$. But $\mathrm{im}(g) = C$ since C is surjective. This means that $C \subseteq \ker(\phi)$. Since trivially $\ker(\phi) \subseteq C$, we have that $\ker(\phi) = C$ which means that ϕ is the 0 map and we shown that $\ker(g_*) = 0$.

Now we want to show that $\operatorname{im}(g_*) = \ker(f_*)$. Suppose that $\phi \in \operatorname{im}(g_a st)$. Then there exists $\psi : C \to N$ such that $\psi \circ g = \phi$. Precomposing with f gives $\psi \circ g \circ f = \phi \circ f$. But $\operatorname{im}(g) = \ker(f)$ means that the left hand side is 0 which means that $\phi \circ f = 0$ and thus $\phi \in \ker(f_*)$.

Suppose that $\phi \in \ker(f_*)$. Then $\phi \circ f = 0$. Define $\psi : C \to N$ by $\psi(c) = \phi(b)$ for any $b \in B$ such that $g(b) = c$. Clearly $\psi \circ g = \phi$. Showing ψ is well defined completes the prove. $b \in B$ always exists for any $c \in C$ since g is surjective. Now suppose that b and b' are both the preimage of c. Then $g(b) = g(b')$ implies $g(b - b') = 0$ which means that $b - b' \in \ker(g)$. But $\ker(g) = \operatorname{im}(f)$ implies $b - b' \in \operatorname{im}(f)$.The first isomorphism theorem tells us that $B/\operatorname{im}(f) \cong C$ since g is surjective. This means that $b - b'$ lie in the same coset of $B/\operatorname{im}(f)$ which means that in this isomorphism b and b' gives the same element c. This means that ψ is well defined. (Self-note: g takes $b \in B$ to $c \in C$ but we know that $C \cong B/\operatorname{im}(f)$ so intrinsically g is well defined in terms of the quotient. The map from C to N is also obvious but we just have to show that ψ makes sense with the quotient)

Now suppose that $0 \to \operatorname{Hom}_R(C, N) \xrightarrow{g_*} \operatorname{Hom}_R(B, N) \xrightarrow{f_*} \operatorname{Hom}_R(A, N)$ is exact. We first show that g is surjective. Pick $N = C/\operatorname{im}(g)$ and take $\psi : C \to C/\operatorname{im}(g)$ to be the quotient map $\psi(c) = c + \operatorname{im}(g)$. For any $b \in B$, we have that $\psi(g(b)) = g(b) + \operatorname{im}(g) = \operatorname{im}(g)$ which means that $\psi \circ g = 0$ which implies that $\psi \in \ker(g_*)$. But g_* being injective means that $\psi = 0$ which means that $\operatorname{im}(g) = C$.

Now we want to show that $\operatorname{im}(f) = \ker(g)$. Take $N = C$. $\operatorname{im}(g_*) = \ker(f_*)$ implies $f_*(g_*(\phi)) = 0$ for all $\phi : C \to N = C$ which means that $\phi \circ g \circ f = 0$. Take ϕ to be the identity map. Then $g \circ f = 0$ and thus $\operatorname{im}(f) \subseteq \ker(g)$,

Now again take $N = B/\operatorname{im}(f)$. Let $\phi : B \to B/\operatorname{im}(f)$ be the projection. Clearly $\phi \circ f$ is the zero map since all of A maps to $\operatorname{im}(f)$ in $B/\operatorname{im}(f)$. This means that $\phi \in \ker(f_*)$. But $\ker(f_*) = \operatorname{im}(g_*)$ means that there exists $\psi : C \to B/\operatorname{im}(f)$ such that $\psi \circ g = \phi$. This means that $\ker(g) \subseteq \ker(\phi)$. But since ϕ is the projection, we have $\ker(\phi) = \operatorname{im}(f)$ which proves that $\ker(g) \subseteq \operatorname{im}(f)$.
[AM94] $\square$

Theorem 7.1.3. *Let $f : A \to B$ be a ring homomorphism. Let M be an A-module. Let N be a B-module. Then we have the following isomorphism:*

$$\operatorname{Hom}_B(M \otimes_A B, N) \cong \operatorname{Hom}_A(M, N)$$

Proof. Notice that this is well defined since f is a ring homomorphism taking A to B, N is naturally also an A module by restriction of scalars. In particular N is an A module by defining the action on N to be $* : A \times N \to N$ by

$$r * n = f(r) \cdot n$$

where $f(r) \cdot n$ is the action of $f(r) \in B$ on $n \in N$.
Define $(\cdot)^+ : \operatorname{Hom}_B(M \otimes_A B, N) \to \operatorname{Hom}_A(M, N)$ by mapping $u : M \otimes_A B \to N$ to

$$u^+(m) = u(m \otimes 1)$$

Similarly, define $(\cdot)^- : \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(M \otimes_A B, N)$ by mapping $v : M \to N$ by

$$v^-(m \otimes b) = v(m) \cdot b$$

Showing that $(u^+)^- = u$ and $(v^-)^+ = v$ completes the proof.
We have that

$$(u^+)^-(m \otimes b) = u^+(m) \cdot b = u(m \otimes 1) \cdot b$$

Since N is a B module we have that $u(m \otimes 1) \cdot b = u(m \otimes b)$ which means that $(u^+)^- = u$.
We also have that

$$(v^-)^+(m) = v^-(m \otimes 1) = v(m) \cdot 1 = v(m)$$

which also proves that $(v^+)^- = v$. $\square$

Proposition 7.1.4. *Let M be an R-module. Let I be an ideal of R. Then we have*

$$M \otimes_R \frac{R}{I} \cong \frac{M}{IM}$$

[*DF10*] *P.370*

Proof. Consider the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

given by the inclusion map and the projection map. Applying the right exact functor $- \otimes_R M$, we have the following exact sequence:

$$0 \longrightarrow I \otimes_R M \longrightarrow R \otimes_R M \longrightarrow R/I \otimes_R M \longrightarrow 0$$

which simplifies to

$$0 \longrightarrow I \otimes_R M \longrightarrow M \longrightarrow R/I \otimes_R M \longrightarrow 0$$

Now the image of the map $I \otimes_R M$ is precisely IM. The exactness of the sequence implies that

$$\frac{M}{IM} \cong \frac{R}{I} \otimes_R M$$

Since the tensor product is commutative in the sense that $M \otimes_R N \cong N \otimes_R M$, we thus obtain the required result. □

Proposition 7.1.5. *Let $f : A \to B$ be a ring homomorphism. If M is a free A-module of rank n, then $M \otimes_A B$ is a free B-module of rank n.*

Proof. Write $M = \bigoplus_{i=1}^n A$ for some indexing set I. Since tensor products distribute over direct sums, we can perform the distribution n times to obtain

$$\bigoplus_{i=1}^{n} (A \otimes_A B) = \bigoplus_{i=1}^{n} B$$

and so we are done. □

Proposition 7.1.6. *Let M be an finitely generated R-module. Then there exists a free module $\bigoplus_{i=1}^n R$ and a map*

$$\bigoplus_{i=1}^{n} R \to M$$

such that the map is surjective.

Proof. We take the definition of a finitely generated R-module as: there exists $a_1, \dots, a_n \in M$ such that for all $x \in M$, there exists $r_1, \dots, r_n \in R$ such that $\sum_{k=1}^n r_k a_k$. Now it is easy to see that the module

$$\bigoplus_{k=1}^{n} Ra_k$$

has a surjective map to M simply by $(r_1, \dots, r_n) \mapsto \sum_{k=1}^n r_k a_k$. □

Definition 7.1.7. Trivial Extension Let R be a ring and M an R-module. Define the trivial extension of R by M to be the additive group $R \otimes M$ together with multiplication defined as $(r, x)(s, y) = (rs, ry + sx)$ for $r, s \in R$ and $x, y \in M$. This ring is denoted as $R \ltimes M$. [Kun86]

Proposition 7.1.8. *Let R be a ring and I an ideal of R. Let m be a maximal ideal. If m does not contain I then $I_m = R_m$ both as localization of R-modules. If m contains I, then $I_m \neq R_m$.*

Proof. Suppose that m does not contain I. Since m is a maximal ideal, R_m is a local ring with maximal ideal m. Take $i \in I$ such that $i \notin m$. This is possible since m does not contain I. Then since m is the unique maximal ideal of R_m, i must be a unit. This means that I_m contains a unit. Since I is an ideal of R we have I_m is an ideal of R_m since localization commutes with quotients. Any ideal that contains a unit is the whole ring and thus we have that $I_m = R_m$.
Now suppose that $I \subseteq m$. Suppose that $I_m = R_m$. Since $1 \in I_m$ we must have $1 = r/s$ for some $r \in I$ and $s \in R \setminus m$. By definition of equality, there must exists some $t \in R \setminus m$ such that $ts - ti = 0$ where ts and $ti \in I$. Now since $R \setminus m$ is a multiplicative set, we have that $t, s \in R \setminus m$ implies $ts \in R \setminus m$. Then this means that $ti \in R \setminus m$. A contradiction since this means $ti \notin I$ even though $ti \in I$ by definition of an ideal. □

7.2 Transcendental Field Extensions

Recall what it means for a field extension to be transcendental. Most of this section refers to [Eis07].

Definition 7.2.1. Transcendental Field Extensions Let L/K be a field extension. We say that L/K is a transcendental field extension if there exists an element $x \in L$ such that x is transcendental over K. In other words, x does not satisfy any univariate polynomial with coefficients in K.

Similar to the basis of vector spaces, we can also define a basis for transcendental field extensions. As one can see, transcendental means that no polynomial relation is satisfied. Thus the concept of linear independence should also be defined in a similar fashion. This leads to the notion of algebraic independence.

Definition 7.2.2. Algebraic Independence Let L/K be a field extension. We say that a subset B of L is algebraically independent over K if the elements of B do not satisfy any non-trivial polynomial relations with coefficients in K.

This definition is slightly different from the one given in [Eis07]. But it is more intuitive to define it this way. One can also show that the definition in [Eis07] and the one above are equivalent, which we will omit here.

Definition 7.2.3. Transcendence Basis Let L/K be a field extension. A transcendence basis of L/K is a subset B of L such that B is algebraically independent and $L/K(B)$ is an algebraic extension.

Indeed if $S/K(B)$ is an algebraic extension, it means that we can no longer add any transcendence elements to our set B, so that B is maximally algebraically independent. We will again, omit the proof here that any two transcendence basis have the same cardinality.

7.3 Filters, Ultrafilters and Principal Filters

Filters often appear in more set theoretic subjects such as topology, set theory and algebra. Ultrafilters, one specific type of filter is used to form the ultraproduct of a collection of algebraic structures so that a lot of weird things will occur that will not appear when one considers only the prototypical examples.

Definition 7.3.1. Filters Let X be a set. A filter $\mathcal{F}$ of X is a family of subsets of X such that

- $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$
- If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$

The idea of a filter is to think of the collection $\mathcal{F}$ of subsets of X as the collection of all large subsets of X. Indeed the third condition shows that any larger subset of subset in $\mathcal{F}$ must also lie in $\mathcal{F}$.

Definition 7.3.2. Ultrafilters Let X be a set. An ultrafilter on X is a filter $\mathcal{F}$ on X such that if $A \subseteq X$ then either A or $X \setminus A$ is an element of $\mathcal{F}$.

Intuitively, ultrafilters on a set X is a maximal filter on the set X. This idea is precisely characterized by the condition that at least one of $A \subseteq X$ and its complement must lie in the filter.

Lemma 7.3.3. *Let X be a set. Let $\mathcal{F}$ be an ultrafilter. If $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

Proof. Suppose for a contradiction that both F and G are not in $\mathcal{F}$. Then by the property of ultrafilter, $X \setminus F$ and $X \setminus G$ are in $\mathcal{F}$. Then by property 2 of a filter, $(X \setminus F) \cap (X \setminus G) \in \mathcal{F}$. This means that $X \setminus (F \cup G) \in \mathcal{F}$. But $X \setminus (F \cup G) \cap (F \cup G) = \emptyset \in \mathcal{F}$ by the same property so we have reached a contradiction. □

Principal filters are essentially the smallest filter containing a chosen subset.

Definition 7.3.4. Principal Filters Let X be a set. A principal filter on X is a filter of the form

$$\mathcal{F} = \{A \in P(X) \mid A \supseteq S\}$$

for a fixed subset S of X.

Lemma 7.3.5. *Let X be a set. Then a principal ultrafilter on X is precisely a filter of the form*

$$\mathcal{F} = \{A \in P(X) \mid A \supseteq \{p\}\}$$

for some $p \in X$.

Proof. It is clear that a filter of the above form is an ultrafilter since $\{p\} \in \mathcal{F}$ implies that for any $S \subseteq X$, either $p \in S$ or $p \in X \setminus S$ so that at least one of S and $X \setminus S$ lie in$\mathcal{F}$. It is also clearly a principal filter by definition.

Now suppose that $\mathcal{G}$ is an arbitrary principal ultrafilter. Then

$$\mathcal{G} = \{A \in P(X) \mid A \supseteq S\}$$

for some fixed subset S of X. I claim that $S = \bigcap_{T \in \mathcal{G}} T$. Clearly all $T \in \mathcal{F}$ are such that $S \subseteq T$ so we have $S \subseteq \bigcap_{T \in \mathcal{G}} T$. Now since $S \in \mathcal{G}$, we also have

$$\bigcap_{T \in \mathcal{G}} T = \left(\bigcap_{T \in \mathcal{G} \setminus \{S\}} T \right) \cap S \subseteq S$$

and so we have equality. It remains to show that S is a singleton. If S is not a singleton, then $S = B \amalg C$ where neither B nor C are empty. In particular, B and C are not in $\mathcal{G}$ since $B, C \subset S$. By property of an ultrafilter, $X \setminus B \in \mathcal{G}$. By property 2 of a filter, we have that $A \cap (X \setminus B) = C \in \mathcal{G}$, which is a contradiction. □

Finally, we note that only principal filters can have sets in the filter that are finite.

Proposition 7.3.6. *Let X be a set. Then an ultrafilter $\mathcal{F}$ on X is a principal filter if and only if it contains finite sets.*

Proof. Suppose that $\mathcal{F}$ is a principal ultrafilter. Then $\mathcal{F}$ clearly contains a finite set. Conversely, suppose that $\mathcal{F}$ is an ultrafilter that contains a finite set S. Then apply 7.3.3 on singleton subsets of S a finite number of times to obtain that a singleton must be in $\mathcal{F}$. □

7.4 Supplement to Example 4.1.7

We provide a proof that given a countable set of points in $\mathbb{R}$, there exists a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ hits them over $\mathbb{N}$.

We have seen that bump functions can be used to create smooth functions.

Lemma 7.4.1. *Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $a, b \in \mathbb{R}$. Then exists a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f((-\infty, x_1]) = a$ and $f([x_2, \infty)) = b$.*

Proof. Without loss of generality, we may assume that $a = 0$ since we can translate the function up by a and construct a smooth function starting at height 0 and reaching height $b - a$. By a similar reasoning, we can scale the function so that without loss of generality, we start at $x_1 = 0$ and $x_2 = 1$.

From MA3H5, we have seen the smooth function

$$g(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{g(x)}{g(x) + g(1-x)}$$

This function is smooth because g is smooth and g is non-zero at the denominator. Also, for $x \leq 0$, we have that $f(x) = 0$ because $g(x) = 0$ and $g(1-x) \neq 0$. We have for $x \geq 1$, $f(x) = 1$ since $g(1-x) = 0$ leaves $f(x) = \frac{g(x)}{g(x)}$ and $g(x) \neq 0$. □

Theorem 7.4.2. *Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $(n_k)_{k\in\mathbb{N}}$ a strictly ascending sequence in $\mathbb{N}$. Then there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(n_k) = y_k$.*

Proof. Let $y_0 = 0$ for convenience. By the above lemma, we can construct smooth functions $f_k : \mathbb{R} \to \mathbb{R}$ such that $f_k((-\infty, n_{k-1}]) = 0$ and $f_k([n_k, \infty)) = y_k - y_{k-1}$. Now define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} f_i(x)$$

We show that this function is smooth by showing that for all $x \in \mathbb{R}$, x has a neighbourhood such that f in that neighbourhood is smooth. (This is reminiscent to defining smooth functions on manifolds, in our case, the "charts" are neighbourhoods). Notice that since $(n_k)_{k\in\mathbb{N}}$ is a strictly ascending infinite sequence, there exists $n_t \in \mathbb{N}$ such that $x < n_t$. Now for any $y \in (-\infty, n_t)$, we have that $f_s(y) = 0$ for all $s \geq t$ by construction. This means that in the domain $(-\infty, n_t)$, f becomes a finite sum

$$f(x) = \sum_{i=1}^{n_t} f_i(x)$$

Since each $f_k(x)$ is smooth, f is smooth in this neighbourhood of x. Thus for all $x \in \mathbb{R}$ there is a neighbourhood of x for which f is smooth.

Now for any k, we have that

$$f(n_k) = \sum_{i=1}^{k} f_i(n_k) = y_k - y_0 = y_k$$

and so we conclude. □